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The Complementary Theorem.[*]

By J. C. Fields.

Let

$$f(z, u) = u^n + f_{n-1}u^{n-1} + \ldots + f_0 = 0 \qquad (1)$$

be an algebraic equation in which the coefficients f_i are rational functions of z, integral or fractional. The equation may be reducible or irreducible, but is supposed to be rid of repeated factors. In the neighborhood of a value $z = a$ the functions u defined by the equation will be represented by n series in powers of $z - a$ with exponents which may be integral or fractional, positive or negative, the number of the negative exponents, however, being in any case finite. These n series, or branches of the equation, as we shall call them, group themselves into a number ν of cycles of orders $r_1, r_2, \ldots, r_\nu$ respectively. A cycle of order r consists of r branches developable in integral powers of $(z - a)^{\frac{1}{r}}$ and successively passing into one another on making the circuit of the point $z = a$ in the z-plane. For the application of the preceding remarks to the value $z = \infty$ it is only necessary to replace the element $z - a$ by $\frac{1}{z}$. The order of a cycle is in general 1, and the cycle consists of a single branch whose development involves only integral exponents. A cycle of order 1 we might call a simple cycle. The number of values of the variable z to which correspond other than simple cycles is finite.

If now in a rational function $H(z, u)$ we substitute for u the series which represents a certain branch corresponding to a value $z = a$, the result will be representable as a series in powers of $z - a$ $\left(\text{or } \frac{1}{z}\right)$, and the lowest exponent in this series we will call the order of coincidence of the function with the branch in question. The order of coincidence of a rational function of (z, u) with a branch of the equation $f(z, u) = 0$ corresponding to a given value of the variable z may be positive or negative; in general, however, it will be 0. Evidently the orders of coincidence of a rational function of (z, u) with different branches of the same cycle will all be equal. The number of cycles with whose branches the orders of coincidence of a given rational function of (z, u) are different from 0 is finite, and the aggregate sum of the orders of coincidence of this function with all the branches of all these cycles is equal to 0. We might then say, for conciseness, that the aggregate sum of all the orders of coincidence of any given rational function of (z, u) with all the branches of all the cycles corresponding to all values of the variable z, the value $z = \infty$ included, is equal to 0. The orders of coincidence of a rational function of (z, u) with the branches of the r cycles corresponding to a given value $z = a$ will evidently be integral multiples of the corresponding fractions $\frac{1}{\nu_1}, \frac{1}{\nu_2}, \ldots, \frac{1}{\nu_r}$ respectively.

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[*] A paper presented to section A at the Dublin meeting of The British Association for the Advancement of Science, September, 1908.

We shall call a system of sets of numbers $\tau_1^{(k)}, \tau_2^{(k)}, \ldots, \tau_{r_k}^{(k)}$ associated with the different values of the variable $z = a_k$, and multiples of the corresponding fractions $\frac{1}{\nu_1^{(k)}}, \frac{1}{\nu_2^{(k)}}, \ldots, \frac{1}{\nu_{r_k}^{(k)}}$, a basis of coincidences for the building of a rational function of (z, u). We conceive a set of numbers to be associated with each value of the variable, the numbers however being all 0 save in the sets associated with a finite number of values of z. When therefore we specify the sets associated with a finite number of values of the variable only, it is to be understood that the sets associated with the remaining values of the variable are all made up of zeros. By the most general rational function built on a given basis of coincidences we mean the most general rational function whose orders of coincidence with the branches of the several cycles corresponding to the various values of z do not fall short of the corresponding numbers mentioned in the basis. It might happen that the most general function built on a proposed basis is identically 0, in which case we call the basis an impossible one. To briefly designate the basis of coincidences here in question we employ the notation (τ).

A basis $(\bar{\tau})$ we say is complementary to the basis (τ) with regard to the level furnished by a rational function $H(z, u)$ when its numbers $\bar{\tau}_1^{(k)}, \bar{\tau}_2^{(k)}, \ldots, \bar{\tau}_{r_k}^{(k)}$ are connected with the numbers $\tau_1^{(k)}, \tau_2^{(k)}, \ldots, \tau_{r_k}^{(k)}$ by the equations

$$\tau_s^{(k)} + \bar{\tau}_s^{(k)} = m_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}} \qquad (s = 1, 2, \ldots, r_k) \qquad (2)$$

functions of (z, u) are rational functions of (z, v), and conversely; orders of coincidence are unchanged by the transformation, the cycles remain the same, and complementary bases are still complementary bases. When then we have proved the complementary theorem for the equation $F(z, v) = 0$, we have evidently at the same time proved it also for the equation $f(z, u) = 0$. In my book* on the algebraic functions I have given a proof of the complementary theorem which involves a number of more or less complicated formulæ† and certain details which can be dispensed with on introducing modifications in the reasoning there employed. The object of the present paper is to give the modified proof here referred to. Before proceeding with the proof, however, it will be necessary to recall certain results obtained in the earlier half of my book, and to make reference to the notation which I have employed.

Starting out from the integral algebraic equation $F(z, v) = 0$, we shall consider rational functions of (z, v). Any such function can be written in the form

$$H(z, v) = h_{n-1}v^{n-1} + \cdots + h_0, \tag{7}$$

where the coefficients h_s are rational functions of z. This is called the reduced form of the function, and we shall suppose in this paper that a rational function is always expressed in its reduced form excepting where the context [illegible] the contrary.

The orders of coincidence of the rational function $H(z, v)$ with the [illegible] of the r_κ cycles corresponding to a value $z = a_\kappa$ we indicate [illegible] $\mu_1^{(\kappa)}, \mu_2^{(\kappa)}, \ldots, \mu_{r_\kappa}^{(\kappa)}$ respectively, and by the symbols $\mu_1^{(\infty)}, \mu_2^{(\infty)}$ [illegible] ively its orders of coincidence with the branches of the r_∞ [illegible] to the value $z = \infty$. If the orders of coincidence of a [illegible] with the branches of the r_κ cycles corresponding to a gi[illegible] do not fall short of the numbers

$$\mu_1^{(\kappa)} = -1 + \frac{1}{\nu_1^{(\kappa)}}, \ldots, \mu_{r_\kappa}^{(\kappa)} = -1 + \frac{1}{\nu_{r_\kappa}^{(\kappa)}} \tag{8}$$

respectively, we say that the function is adjoint relatively to the equation $F(z, v) = 0$ for the value of the variable in question, and the coincidences are said to constitute a set of adjoint orders of coincidence. This definition extends for all finite values of the variable $z = a_k$, and for the value $z = \infty$ by the equations

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* Theory of the Algebraic Functions of a Complex Variable: Mayer & Müller, Berlin, 1906. In the present paper this book will be cited under the letters A. F.

† A. F., Chapter XI. The proof here given displaces the greater part of Chapter XI and modifies the beginning of the chapter following.

$$\tau_s^{(\infty)} + \bar{\tau}_s^{(\infty)} = m_s^{(\infty)} + 1 + \frac{1}{\nu_s^{(\infty)}} \quad (s = 1, 2, \ldots, r_\infty), \tag{3}$$

where $m_1^{(k)}, m_2^{(k)}, \ldots, m_{r_k}^{(k)}$ represent the actual orders of coincidence of the rational function $K(z, u)$ corresponding to the value $z = a_k$. The basis (τ) is then evidently also complementary to the basis $(\bar{\tau})$ with regard to the levels furnished by the function $K(z, u)$.

On employing the symbols N_τ and $N_{\bar{\tau}}$ to designate the numbers of arbitrary constants involved in the expressions for the most general rational functions on the complementary bases (τ) and $(\bar{\tau})$, respectively, the statement of the complementary theorem is contained in the equation

$$N_\tau + \tfrac{1}{2} \sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)} = N_{\bar{\tau}} + \tfrac{1}{2} \sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)} \nu_s^{(k)}, \tag{4}$$

where the summations with regard to k are supposed to extend to all values of the variable z, the value $z = \infty$ included.

In the definition of complementary bases given above, it would at first sight appear as if the value $z = \infty$ had been treated differently from the finite values of the variable. The difference in treatment is however only apparent, and the complementary bases (τ) and $(\bar{\tau})$ could be symmetrically defined with regard to all values of the variable z, the value $z = \infty$ included, by the system of equations

$$\tau_s^{(k)} + \bar{\tau}_s^{(k)} = \bar{m}_s^{(k)} + \epsilon^{(k)} - 1 + \frac{1}{\nu_s^{(k)}} \quad (s = 1, 2, \ldots, r_k), \tag{5}$$

where the numbers $\epsilon^{(k)}$ are all zero, save a finite number, which are integers positive or negative, subject only to the condition that $\sum_k \epsilon^{(k)} = 2$; and where the numbers $\bar{m}_1^{(k)}, \bar{m}_2^{(k)}, \ldots, \bar{m}_{r_k}^{(k)}$ represent the actual orders of coincidence of a rational function $\bar{K}(z, u)$ which is connected with the function $K(z, u)$ by the relation

$$K(z, u) = \prod_k (z - a_k)^{\epsilon^{(k)}} \cdot \bar{K}(z, u),$$

where the product is extended to all finite values of the variable $z = a_k$.

By a transformation $v = gu$, where g is an integral rational function of z, we can make the equation $f(z, u) = 0$ go over into an equation

$$F(z, v) = v^n + F_{n-1} v^{n-1} + \cdots + F_0 = 0 \tag{6}$$

in which the coefficients F_i are integral rational functions of z. Rational not only to all finite values of the variable, but also to the value $z = \infty$. If the product of two functions is adjoint for a given value of the variable, we say that the functions are complementary adjoint for the value of the variable in question; and if the orders of coincidence of the product with the branches of the r_k cycles are equal to or greater than the corresponding numbers

$$i + \mu_1^{(k)} - 1 + \frac{1}{\nu_1^{(k)}}, \ldots, i + \mu_{r_k}^{(k)} - 1 + \frac{1}{\nu_{r_k}^{(k)}}, \tag{9}$$

where i is any positive integer, we say that the functions are complementary adjoint to the order i.

In considering a rational function of (z, v) with reference to the cycles corresponding to the value $z = \infty$, it is at times convenient to transform to the variables $\xi = z^{-1}$, $\eta = z^{-m} v$, where z^m is the highest power of z which appears in the equation $F(z, v) = 0$. This equation then goes over into the integral algebraic equation

$$G(\xi, \eta) = 0, \tag{10}$$

in which η is the dependent variable. The rational function

$$G'_\eta(\xi, \eta) = \xi^{m(n-1)} F'_v(z, v)$$

evidently has, as orders of coincidence with the branches of the cycles corresponding to the value $\xi = 0$, the numbers

$$m(n-1) + \mu_1^{(\infty)}, \ldots, m(n-1) + \mu_{r_\infty}^{(\infty)}; \tag{11}$$

and the orders of coincidence defining adjointness relatively to the equation $G(\xi, \eta) = 0$ for the cycles corresponding to the value $\xi = 0$, are

$$m(n-1) + \mu_1^{(\infty)} - 1 + \frac{1}{\nu_1^{(\infty)}}, \ldots, m(n-1) + \mu_{r_\infty}^{(\infty)} - 1 + \frac{1}{\nu_{r_\infty}^{(\infty)}} \tag{12}$$

respectively. When the product of two rational functions has orders of coincidence for the value $\xi = 0$ which do not fall short of these numbers, they are complementary adjoint relatively to the equation $G(\xi, \eta) = 0$ for the value of ξ here in question; and when the orders of coincidence of the product do not fall short of the numbers

$$m(n-1) + i + \mu_1^{(\infty)} - 1 + \frac{1}{\nu_1^{(\infty)}}, \ldots, m(n-1) + i + \mu_{r_\infty}^{(\infty)} - 1 + \frac{1}{\nu_{r_\infty}^{(\infty)}} \tag{13}$$

respectively, the functions are complementary adjoint to the order i relatively to the equation (10) for the value $\xi = 0$.

A rational function of (z, v) can be written in the form

$$H(z, v) = P(z, v) + \sum_k \frac{\phi^{(i_k)}(z, v)}{(z - a_k)^{i_k}}, \tag{14}$$

where the expressions $P(z, v)$ and $\phi^{(i_k)}(z, v)$ represent polynomials in (z, v), reduced with regard to v, and where a polynomial $\phi^{(i_k)}(z, v)$ is of degree less than i_k in z. Here the summation is supposed to extend to all those finite values $z = a_k$ for which the function $H(z, v)$ may become infinite, as also to all those values of the variable to which correspond multiple points on the curve $F(z, v) = 0$. Transformed to terms of (ξ, η) our rational function can also be represented in the form

$$H(z, v) = -\frac{\phi^{(i_\infty + j)}(\xi, \eta)}{\xi^{i_\infty}} + \xi^j((\xi, \eta)), \tag{15}$$

where $\phi^{(i_\infty + j)}(\xi, \eta)$ is a polynomial in (ξ, η) reduced in η, and of a degree in ξ which is less than $i_\infty + j$. Here j is an integer which is at our disposal, and the expression $((\xi, \eta))$ signifies a reduced polynomial in η with coefficients which, expanded in powers of ξ, involve no negative exponents.

If now we characterize the function $H(z, v)$ by saying that it is to have orders of coincidence with the branches of the r_∞ cycles corresponding to the value $z = \infty$, which do not fall short of the numbers $\tau_1^{(\infty)}, \tau_2^{(\infty)}, \ldots, \tau_{r_\infty}^{(\infty)}$ respectively, while it is to become infinite to orders not exceeding $\sigma_1^{(k)}, \sigma_2^{(k)}, \ldots, \sigma_{r_k}^{(k)}$ respectively for the branches of the r_k cycles corresponding to a value $z = a_k$, we shall have for any exponent i_k in formula (14) the greatest of the integers*

$$[\mu_1^{(k)} + \sigma_1^{(k)}], \ldots, [\mu_{r_k}^{(k)} + \sigma_{r_k}^{(k)}], \tag{16}$$

and for the exponent i_∞ in formula (15) the greatest of the integers†

$$m(n-1) + [\mu_1^{(\infty)} + \sigma_1^{(\infty)}], \ldots, m(n-1) + [\mu_{r_\infty}^{(\infty)} + \sigma_{r_\infty}^{(\infty)}], \tag{17}$$

where corresponding numbers in the sets $\sigma_1^{(\infty)}, \ldots, \sigma_{r_\infty}^{(\infty)}$ and $\tau_1^{(\infty)}, \ldots, \tau_{r_\infty}^{(\infty)}$ are equal and of opposite signs excepting where a number τ is positive, in which case the corresponding number σ has the value 0.

It is to be understood that the integer j in formula (15) is not negative and furthermore we shall find it convenient to assume that it is not less than the greatest of the numbers $\tau_1^{(\infty)}, \tau_2^{(\infty)}, \ldots, \tau_{r_\infty}^{(\infty)}$, so that the coefficients in the expression $\xi^j((\xi, \eta))$ will not be conditioned by the orders of coincidence of $H(z, v)$ for the value $z = \infty$. A numerator $\phi^{(i_k)}(z, v)$ appearing in the summation in formula (14) will have orders of coincidence with the branches of the r_k corresponding cycles which do not fall short of the numbers

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* *A. F.*, X.

† *A. F.*, XI, 6.

$$i_k - \sigma_1^{(k)},\ i_k - \sigma_2^{(k)},\ \ldots,\ i_k - \sigma_{r_k}^{(k)} \tag{18}$$

respectively, and will involve ρ_k arbitrary constants, where*

$$\rho_k = \sum_{s=1}^{r_k} \sigma_s^{(k)} \nu_s^{(k)} + \tfrac{1}{2}\sum_{s=1}^{r_k}\left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}\right)\nu_s^{(k)}. \tag{19}$$

The numerator $\phi^{(i_\infty + j)}(\xi, \eta)$ appearing on the right-hand side of formula (15) will have orders of coincidence with the branches of the r_∞ cycles corresponding to the value $z = \infty$, which do not fall short of the numbers

$$i_\infty + \tau_1^{(\infty)},\ i_\infty + \tau_2^{(\infty)},\ \ldots,\ i_\infty + \tau_{r_\infty}^{(\infty)} \tag{20}$$

respectively, and will involve ρ_∞ arbitrary constants, where†

$$\rho_\infty = \sum_{s=1}^{r_\infty} (j - \tau_s^{(\infty)})\nu_s^{(\infty)} + \tfrac{1}{2}\sum_{s=1}^{r_\infty}\left\{m(n-1) + \mu_s^{(\infty)} - 1 + \frac{1}{\nu_s^{(\infty)}}\right\}\nu_s^{(\infty)}. \tag{21}$$

We can then write

$$\phi^{(i)}(z, v) = \delta_1^{(k)}\phi_1^{(i_k)}(z, v) + \ldots + \delta_{\rho_k}^{(k)}\phi_{\rho_k}^{(i_k)}(z, v), \tag{22}$$

$$\phi^{(i_\infty + j)}(\xi, \eta) = \delta_1^{(\infty)}\phi_1^{(i_\infty + j)}(\xi, \eta) + \ldots + \delta_{\rho_\infty}^{(\infty)}\phi_{\rho_\infty}^{(i_\infty + j)}(\xi, \eta), \tag{23}$$

where the δ's are arbitrary constants excepting for the conditions imposed upon them in assuming the function $H(z, v)$ to be simultaneously representable in the forms (14) and (15). Equating these two forms we have

$$P(z, v) + \sum_k \sum_{s=1}^{\rho_k} \frac{\delta_s^{(k)}\phi_s^{(i_k)}(z, v)}{(z - a_k)^{i_k}} = -\sum_{s=1}^{\rho_\infty} \frac{\delta_s^{(\infty)}\phi_s^{(i_\infty + j)}(\xi, \eta)}{\xi^{i_\infty}} + \xi^j((\xi, \eta)). \tag{24}$$

Here the polynomial $P(z, v)$ identifies itself with the integral portion of the expression $\xi^{-i_\infty}\phi^{(i_\infty + j)}(\xi, \eta)$ transformed to terms of (z, v) and therefore depends on the constants $\delta_s^{(\infty)}$. Furthermore, the totality of the conditions to which the constants δ must be subjected is obtained on transforming

$$-\xi^{-i_\infty}\sum_{s=1}^{\rho_\infty}\delta_s^{(\infty)}\phi_s^{(i_\infty + j)}(\xi, \eta) \tag{25}$$

to terms of (z, v) and identifying the terms in the transformed expression which involve negative powers of z, with the corresponding terms in the expression

$$\sum_k \sum_{s=1}^{\rho_k} \frac{\delta_s^{(k)}\phi_s^{(i_k)}(z, v)}{(z - a_k)^{i_k}} \tag{26}$$

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* *A. F.*, X, 10.

† *A. F.*, XI, 10.

after the coefficients of the several powers of v have been developed in powers of $\frac{1}{z}$. The developments here in question will evidently involve only negative powers of z. We desire then to study the conditions imposed on the constants δ by the statement of the identity

$$\sum_k \sum_{s=1}^{\rho_k} \frac{\delta_s^{(k)} \phi_s^{(\iota_k)}(z, v)}{(z-a_k)^{\iota_k}} + \sum_{s=1}^{\rho_\infty} \frac{\delta_s^{(\infty)} \phi_s^{(\iota_\infty + j)}(\xi, \eta)}{\xi^{\iota_\infty}} = \text{Polynomial in } (z, v) + \xi^j((\xi, \eta)). \quad (27)$$

We have to suppose that the total expression on the left-hand side of this identity has been transformed to terms of (z, v), and that it is then represented as a polynomial in v with coefficients which are developed in powers of $\frac{1}{z}$, the number of negative powers of $\frac{1}{z}$ involved being of course finite. The conditions to which the constants δ are subjected are then obtained on equating to 0 the aggregate coefficient of an element $z^{-r} v^{n-t}$, where t can have any one of the values $1, \ldots, n$ and where in $z^{-r} v^{n-t} = \xi^{r-m(n-t)} \eta^{n-t}$ the integer r is positive and subject also to the condition $r - m(n-t) < j$.

In Chapter IX of my book I have introduced a certain polynomial $R(z, v)$, such that its product by the function $F(z, v)$ has identically the form

$$F(z, v) \,.\, R(z, v) = v^{2n-1} + S_{n-1} v^{n-1} + \ldots + S_0. \quad (28)$$

Here the coefficient of v^{2n-1} in the product requires the coefficient of v^{n-1} in $R(z, v)$ to be 1, and the function $R(z, v)$ is then completely determined by the vanishing of the coefficients of $v^{2n-2}, \ldots, v^n$ in the product. The coefficients $S_{n-1}, \ldots, S_0$ in the product are of course completely determined when $R(z, v)$ is determined. At the same time also we evidently have 1 for the coefficient of η^{n-1} in the function

$$\overline{R}(\xi, \eta) = \xi^{m(n-1)} R(z, v), \quad (29)$$

and terms involving the powers $\eta^{2n-2}, \ldots, \eta^n$ are lacking in the product $G(\xi, \eta) \,.\, \overline{R}(\xi, \eta)$.

Consider all the polynomials $\phi(z, v)$, reduced in v, which possess a certain set of adjoint orders of coincidence $\mu_1', \ldots, \mu_r'$ corresponding to a value $z = a$. Supposing these functions to be arranged according to powers of $z - a$ and v and discarding terms divisible by $(z-a)^\iota$, we obtain a system of functions of the type $\phi^{(\iota)}(z, v)$, and of these a certain number ρ will be linearly independent of one another. A complete system of linearly independent functions of this type we indicate by the notation

$$\phi_s^{(i)}(z, v) \qquad (s = 1, 2, \ldots, \rho). \tag{30}$$

Multiply these functions by the function $R(z, v)$ and write the products in the form

$$R(z, v) \,.\, \phi_s^{(i)}(z, v) = \theta_s(z, v) + v^{n-1}\zeta_s^{(i)}(z, v) \qquad (s = 1, 2, \ldots, \rho), \tag{31}$$

where in the element $v^{n-1}\zeta_s^{(i)}(z, v)$ are included all those terms of the product which are divisible by v^{n-1} and which are not at the same time divisible by $(z-a)^i$. Any term then in $\theta_s(z, v)$ which is divisible by v^{n-1} must also be divisible by $(z-a)^i$. At the same time we also have

$$F(z, v) \,.\, \zeta_s^{(i)}(z, v) = \Pi_s(z, v) + v^n \phi_s^{(i)}(z, v) \qquad (s = 1, 2, \ldots, \rho), \tag{32}$$

where $v^n \phi_s^{(i)}(z, v)$ is constituted out of all those terms in the product on the left which are divisible by v^n and which are not divisible by $(z-a)^i$. The truth of this is seen on multiplying the identities (31) through by $F(z, v)$ and equating on the two sides those terms which are divisible by v^{2n-1} but not by $(z-a)^i$, bearing the while in mind the formula (28).

In like manner we might consider all those polynomials $\phi(\xi, \eta)$, reduced in η, which possess a certain set of adjoint orders of coincidence relative to the equation $G(\xi, \eta) = 0$ for the value $\xi = 0$. On discarding in these polynomials terms divisible by ξ^i we obtain a system of functions of the type $\phi^{(i)}(\xi, \eta)$, and of these a certain number ρ_∞ will be linearly independent of one another. A complete system of linearly independent functions of this type we might indicate by the notation

$$\phi_s^{(i)}(\xi, \eta) \qquad (s = 1, 2, \ldots, \rho_\infty). \tag{33}$$

Multiplying these functions by the function $\overline{R}(\xi, \eta)$ we can write the products in the form

$$\overline{R}(\xi, \eta) \cdot \phi_s^{(i)}(\xi, \eta) = \theta_s(\xi, \eta) + \eta^{n-1}\zeta_s^{(i)}(\xi, \eta) \qquad (s = 1, 2, \ldots, \rho_\infty), \tag{34}$$

where in the element $\eta^{n-1}\zeta_s^{(i)}(\xi, \eta)$ are included all those terms of the product which are divisible by η^{n-1} and which are not at the same time divisible by ξ^i. Any term then in $\theta_s(\xi, \eta)$ which is divisible by η^{n-1} must also be divisible by ξ^i. At the same time we also have

$$G(\xi, \eta) \cdot \zeta_s^{(i)}(\xi, \eta) = \Pi_s(\xi, \eta) + \eta^n \phi_s^{(i)}(\xi, \eta) \qquad (s = 1, 2, \ldots, \rho_\infty), \tag{35}$$

where $\eta^n \phi_s^{(i)}(\xi, \eta)$ is constituted out of all those terms in the product on the left which are divisible by η^n and which are not divisible by ξ^i. Relatively to the equation $G(\xi, \eta) = 0$, and for the value $\xi = 0$, formulae (34) and (35) are the analogues of the formulae (31) and (32), respectively, for the equation $F(z, v) = 0$ relatively to the value $z = a$. When we make use of the expression $\theta_s(\xi, \eta)$, for example, in the one set of formulae, it will be understood of course that we are not dealing with the same form as that which was designated by $\theta_s(z, v)$ in the other set of formulae.

Now the necessary and sufficient conditions in order that an integral rational function $\psi(z, v)$ may be complementary adjoint to the order i to a system of integral rational functions $\phi(z, v)$, whose orders of coincidence with the branches of the several cycles corresponding to the value $z = a$ are not less than the respective members of a set of adjoint numbers $\mu_1', \ldots, \mu_r'$, are obtained* on equating to 0 the coefficients of $(z - a)^{i-1} v^{n-1}$ in the ρ products

$$\zeta_s^{(i)}(z, v) \,.\, \psi(z, v) \qquad (s = 1, 2, \ldots, \rho). \tag{36}$$

This is evidently equivalent to saying that the necessary and sufficient conditions in order that the orders of coincidence of the integral rational function $\psi(z, v)$ may be complementary adjoint to the orders of coincidence of the system of functions $(z - a)^{-i} \phi(z, v)$, are obtained on equating to 0 the coefficients of $(z - a)^{-1} v^{n-1}$ in the ρ products

$$(z - a)^{-i} \zeta_s^{(i)}(z, v) \,.\, \psi(z, v) \qquad (s = 1, 2, \ldots, \rho). \tag{37}$$

The coefficients here in question are however the residues of the coefficients of v^{n-1} relative to the value $z = a$ in the ρ products (37). Now in the rational functions of z which constitute the coefficients of the several powers of v in these products the only poles which can present themselves are evidently $z = a$ and $z = \infty$, and the only residues which appear must correspond to these values. Since, however, the sum of the residues of a rational function of z is zero, if the residues of the coefficients of v^{n-1} in the products (37) relative to the value $z = a$ vanish, so also will the residues relative to the value $z = \infty$ vanish, and conversely.

We may then say that the necessary and sufficient conditions in order that the orders of coincidence of an integral rational function $\psi(z, v)$ for the value $z = a$ may be complementary adjoint to the orders of coincidence of the system of functions $(z - a)^{-i} \phi(z, v)$, in which the numerator $\phi(z, v)$ possesses a certain system of adjoint orders of coincidence, are obtained on equating to 0 the residues relative to the value $z = \infty$ of the coefficients of v^{n-1} in the ρ products (37).

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* *A. F.*, IX.

In like manner we see that the necessary and sufficient conditions in order that the orders of coincidence of an integral rational function $\overline{\psi}(\xi, \eta)$, relative to the equation $G(\xi, \eta) = 0$ for the value $\xi = 0$, may be complementary adjoint to the orders of coincidence of the system of functions $\xi^{-(i_\infty + j)} \phi(\xi, \eta)$, in which the numerator $\phi(\xi, \eta)$ possesses the system of adjoint orders of coincidence $i_\infty + \tau_1^{(\infty)}, \ldots, i_\infty + \tau_{r_\infty}^{(\infty)}$, are obtained on equating to 0 the residues relative to the value $\xi = 0$ (or $\xi = \infty$) of the coefficients of η^{n-1} in the ρ_∞ products

$$\xi^{-(i_\infty + j)} \zeta_s^{(i_\infty + j)}(\xi, \eta) . \overline{\psi}(\xi, \eta) \qquad (s = 1, 2, \ldots, \rho_\infty). \tag{38}$$

This is evidently equivalent to the statement that the necessary and sufficient conditions in order that the orders of coincidence of an integral rational function $\overline{\psi}(\xi, \eta)$, relative to the equation $G(\xi, \eta) = 0$ for the value $\xi = 0$, may not be less than the numbers

$$j - \tau_1^{(\infty)} + m(n-1) + \mu_1^{(\infty)} - 1 + \frac{1}{\nu_1^{(\infty)}}, \ldots, j - \tau_{r_\infty}^{(\infty)} + m(n-1) + \mu_{r_\infty}^{(\infty)} - 1 + \frac{1}{\nu_{r_\infty}^{(\infty)}}$$

respectively, are obtained on equating to 0 the coefficients of $\xi^{-1} \eta^{n-1}$ in the ρ_∞ products (38).

Again this statement is equivalent to saying that the necessary and sufficient conditions in order that the orders of coincidence, relative to the equation $G(\xi, \eta) = 0$, of a function

$$\xi^{-j-m(n-1)} \overline{\psi}(\xi, \eta),$$

for the value $\xi = 0$, may not be less than the numbers

$$-\tau_1^{(\infty)} + \mu_1^{(\infty)} - 1 + \frac{1}{\nu_1^{(\infty)}}, \ldots, -\tau_{r_\infty}^{(\infty)} + \mu_{r_\infty}^{(\infty)} - 1 + \frac{1}{\nu_{r_\infty}^{(\infty)}}$$

respectively, are obtained on equating to 0 the coefficients of $z^{-m(n-1)+1} v^{n-1}$ in the ρ_∞ products (38) transformed to terms of (z, v). We conclude therefore that the necessary and sufficient conditions in order that the orders of coincidence of a function $\xi^{-j-m(n-1)} \overline{\psi}(\xi, \eta)$ transformed to terms of (z, v), for the value $z = \infty$, may be complementary adjoint relatively to the equation $F(z, v) = 0$, to the orders of coincidence $\tau_1^{(\infty)}, \ldots, \tau_{r_\infty}^{(\infty)}$ of the system of functions $\xi^{-i_\infty} \phi(\xi, \eta)$, are obtained on equating to 0 the coefficients of $z^{-1} v^{n-1}$ in the ρ_∞ products

$$\xi^{-i_\infty} \zeta_s^{(i_\infty + j)}(\xi, \eta) . \xi^{-j-m(n-1)+2} \overline{\psi}(\xi, \eta) \qquad (s = 1, 2, \ldots, \rho_\infty),$$

transformed to terms of (z, v).

By choosing j sufficiently large the expression $\xi^{-j-m(n-1)+2} \overline{\psi}(\xi, \eta)$ will become a polynomial in (z, v), and conversely any polynomial in (z, v) can be written in this form on choosing j sufficiently large.

Writing therefore

$$\xi^{-j-m(n-1)+2}\overline{\psi}(\xi,\eta)=\psi(z,v)$$

a polynomial in (z,v), we derive that the necessary and sufficient conditions, in order that the orders of coincidence of the function $z^2\psi(z,v)$ for the value $z=\infty$ may be complementary adjoint, relatively to the equation $F(z,v)=0$, to the orders of coincidence $\tau_1^{(\infty)},\ldots,\tau_{r_\infty}^{(\infty)}$, are obtained on equating to 0 the coefficients of $z^{-1}v^{n-1}$ in the ρ_∞ products

$$\xi^{-i_\infty}\zeta_s^{(i_\infty+j)}(\xi,\eta)\,.\,\psi(z,v)\qquad(s=1,2,\ldots,\rho_\infty),\tag{39}$$

transformed to terms of (z,v), the integer j being chosen sufficiently large.

We may then say that the necessary and sufficient conditions, in order that the orders of coincidence of the polynomial $\psi(z,v)$ for the value $z=\infty$ may be complementary adjoint to the order 2 to the orders of coincidence $\tau_1^{(\infty)},\ldots,\tau_{r_\infty}^{(\infty)}$ of a system of functions $\xi^{-i_\infty}\phi(\xi,\eta)$, are obtained on equating to 0 the residues relative to the value $z=\infty$ of the coefficient of v^{n-1} in the ρ_∞ products (39), the integer j being chosen large enough.

We shall now return to the identity (27) and to the study of the conditions which its existence imposes on the constants δ. Multiply the identity through by the function

$$R(z,v)=\xi^{-m(n-1)}\overline{R}(\xi,\eta).$$

Replace the products

$$R(z,v)\,.\,\phi_s^{(i_k)}(z,v)$$

by expressions of the type given on the right-hand side of formula (31), and the products

$$\overline{R}(\xi,\eta)\,.\,\phi_s^{(i_\infty+j)}(\xi,\eta)$$

by expressions of the type given on the right-hand side of formula (34), on replacing i by $i_\infty+j$. In the resulting identity retain only those terms which are divisible by v^{n-1} or η^{n-1}, and we evidently obtain an identity of the form

$$v^{n-1}\sum_k\sum_{s=1}^{\rho_k}\frac{\delta_s^{(k)}\zeta_s^{(i_k)}(z,v)}{(z-a_k)^{i_k}}+\xi^{-m(n-1)}\eta^{n-1}\sum_{s=1}^{\rho_\infty}\frac{\delta_s^{(\infty)}\zeta_s^{(i_\infty+j)}(\xi,\eta)}{\xi^{i_\infty}}$$
$$=\text{Product of }v^{n-1}\text{ and a polynomial in }(z,v)+\xi^{-m(n-1)}\eta^{n-1}\xi^{j}((\xi,\eta))_1.$$

Dividing through by $v^{n-1}=\xi^{-m(n-1)}\eta^{n-1}$ this identity becomes

$$\sum_k\sum_{s=1}^{\rho_k}\frac{\delta_s^{(k)}\zeta_s^{(i_k)}(z,v)}{(z-a_k)^{i_k}}+\sum_{s=1}^{\rho_\infty}\frac{\delta_s^{(\infty)}\zeta_s^{(i_\infty+j)}(\xi,\eta)}{\xi^{i_\infty}}$$
$$=\text{Polynomial in }(z,v)+\xi^{j}((\xi,\eta))_1.\tag{40}$$

Now multiply this identity through by the function

$$F(z,v)=\xi^{-mn}G(\xi,\eta).$$

Replace the products

$$F(z, v) \,.\, \zeta_s^{(\iota_k)}(z, v)$$

by expressions of the type given on the right-hand side of formula (32), and the products

$$G(\xi, \eta) \,.\, \zeta_s^{(\iota_\infty + j)}(\xi, \eta)$$

by expressions of the type given on the right-hand side of formula (35). In the resulting identity retain only those terms which are divisible by v^n or η^n, and we arrive at an identity of the form

$$v^n \sum_k \sum_{s=1}^{\rho_k} \frac{\delta_s^{(k)} \phi_s^{(\iota_k)}(z, v)}{(z - a_k)^{\iota_k}} + \xi^{-mn} \eta^n \sum_{s=1}^{\rho_\infty} \frac{\delta_s^{(\infty)} \phi_s^{(\iota_\infty + j)}(\xi, \eta)}{\xi^{\iota_\infty}}$$
$$= \text{Product of } v^n \text{ and a polynomial in } (z, v) + \xi^{-mn+j} \eta^n ((\xi, \eta))_2 .$$

Dividing through by $v^n = \xi^{-mn} \eta^n$ this identity becomes

$$\sum_k \sum_{s=1}^{\rho_k} \frac{\delta_s^{(k)} \phi_s^{(\iota_k)}(z, v)}{(z - a_k)^{\iota_k}} + \sum_{s=1}^{\rho_\infty} \frac{\delta_s^{(\infty)} \phi_s^{(\iota_\infty + j)}(\xi, \eta)}{\xi^{\iota_\infty}} = \text{Polynomial in } (z, v) + \xi^j ((\xi, \eta))_2 .$$

This last identity, at least so far as the constants δ are concerned, is equivalent to the identity (27), and it follows from the identity (40). The identity (40) is however a consequence of the identity (27), and therefore imposes no other conditions on the constants than those which are involved in the identity (27). The conditions imposed on the constants δ by the identities (27) and (40) are therefore precisely the same, and for the study of these conditions we can consequently make use of the identity (40) instead of employing the identity (27).

On the left-hand side of the identity (40) equate to 0 the coefficients of terms in

$$z^{-r} v^{n-t} = \xi^{r-(n-t)m} \eta^{n-t} \qquad (0 < r < j + (n - t) m; \; t = 1, 2, \ldots, n). \quad (41)$$

This gives us a number of linear equations which must be satisfied by the constants δ. Of these a certain number will be independent, and the remaining ones will follow as a consequence of these. Suppose we represent by $\alpha_{r-1, t-1}$ the multipliers of the respective equations which give the linear connection between a dependent equation and the independent equations. Construct the polynomial

$$\psi(z, v) = \sum_{r, t} \alpha_{r-1, t-1} z^{r-1} v^{t-1}, \tag{42}$$

and the dependence of the one equation on the others is evidently equivalent to the statement that the coefficient of $z^{-1} v^{n-1}$ is equal to 0 in each one of the products

$$(z - a_k)^{-\iota_k} \zeta_s^{(\iota_k)}(z, v) \,.\, \psi(z, v) \qquad (s = 1, 2, \ldots, \rho_k),$$
$$\xi^{-\iota_\infty} \zeta_s^{(\iota_\infty + j)}(\xi, \eta) \,.\, \psi(z, v) \qquad (s = 1, 2, \ldots, \rho_\infty).$$

These conditions, however, from what we have seen in the preceding, are those which are necessary and sufficient in order that the polynomial $\psi(z, v)$ may be complementary adjoint to the orders of coincidence $-\sigma_1^{(k)}, -\sigma_2^{(k)}, \ldots, -\sigma_{r_k}^{(k)}$ corresponding to the finite values of the variable $z = a_k$, and in order that it may be complementary adjoint to the order 2 to the orders of coincidence $\tau_1^{(\infty)}, \tau_2^{(\infty)}, \ldots, \tau_{r_\infty}^{(\infty)}$ corresponding to the value $z = \infty$. Furthermore, among the conditions to which we subject the constants δ, the number which are not independent of one another is evidently equal to the number of arbitrary coefficients involved in the most general polynomial

$$\psi(z, v) = \sum_{r, t} a_{r-1, t-1} z^{r-1} v^{t-1} \quad (0 < r < j + m(n-t);\ t = 1, 2, \ldots, n), \quad (43)$$

whose orders of coincidence for the finite values $z = a_k$ do not fall short of the numbers

$$\sigma_1^{(k)} + \mu_1^{(k)} - 1 + \frac{1}{\nu_1^{(k)}}, \ldots, \sigma_{r_k} + \mu_{r_k} - 1 + \frac{1}{\nu_{r_k}^{(k)}} \quad (44)$$

respectively, while its orders of coincidence for the value $z = \infty$ are not less than the numbers

$$-\tau_1^{(\infty)} + \mu_1^{(\infty)} + 1 + \frac{1}{\nu_1^{(\infty)}}, \ldots, -\tau_{r_\infty}^{(\infty)} + \mu_{r_\infty}^{(\infty)} + 1 + \frac{1}{\nu_{r_\infty}^{(\infty)}} \quad (45)$$

respectively. The total number of arbitrary constants involved in the expression of the most general polynomial $\psi(z, v)$ of the character here in question we designate by the symbol N_ψ. The total number of equations to which we subject the constants δ is given by the summation

$$(j-1)n + \sum_{t=1}^{n} m(n-t) = (j-1)n + \tfrac{1}{2} mn(n-1),$$

and the number of linearly independent equations satisfied by the constants δ is therefore

$$(j-1)n + \tfrac{1}{2} mn(n-1) - N_\psi. \quad (46)$$

The total number of the constants δ, as we see on referring to formulae (19) and (21), is

$$\sum_k \rho_k = \tfrac{1}{2} mn(n-1) + jn + \tfrac{1}{2} \sum_k \sum_{s=1}^{r_k} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}\right) \nu_s^{(k)} + \sum_k{}' \sum_{s=1}^{r_k} \sigma_s^{(k)} \nu_s^{(k)} - \sum_{s=1}^{r_\infty} \tau_s^{(\infty)} \nu_s^{(\infty)}, \quad (47)$$

where the accented summation with regard to k is supposed to extend only to finite values $z = a_k$, whereas the summation without the accent extends also to the value $z = \infty$.

We have now come back to the reasoning employed about the middle of Chapter XII of "The Theory of the Algebraic Functions," and from here on we shall follow the text in developing the subject. The number of the constants δ which remain arbitrary on solving the system of equations to which we have subjected them, is obtained on subtracting the number given in formula (46) from that given in formula (47). This gives us the expression

$$N_\psi + n + \tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}\right)\nu_s^{(k)} + \sum_k{}' \sum_{s=1}^{r_k} \sigma_s^{(k)}\nu_s^{(k)} - \sum_{s=1}^{r_\infty} \tau_s^{(\infty)}\nu_s^{(\infty)} \tag{48}$$

for the number of the arbitrary constants δ, and therefore for the number of the independent* arbitrary constants involved in the coefficients of the most general rational function $H(z, v)$ which becomes infinite to orders not exceeding $\sigma_1^{(k)}, \sigma_2^{(k)}, \ldots, \sigma_{r_k}^{(k)}$ respectively, for the branches of the several cycles corresponding to the various finite values of the variable $z = a_k$, while its orders of coincidence with the branches of the several cycles corresponding to the value $z = \infty$ do not fall short of the numbers $\tau_1^{(\infty)}, \tau_2^{(\infty)}, \ldots, \tau_{r_\infty}^{(\infty)}$ respectively. On employing the symbol N_H to designate the number of arbitrary constants involved in the expression of the most general function $H(z, v)$ of the character here in question, and on replacing the symbols $-\sigma_1^{(k)}, -\sigma_2^{(k)}, \ldots, \sigma_{r_k}^{(k)}$ in the expression (48) by the symbols $\tau_1^{(k)}, \tau_2^{(k)}, \ldots, \tau_{r_k}^{(k)}$, we have

$$N_H = N_\psi + n + \tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}\right)\nu_s^{(k)} - \sum_k \sum_{s=1}^{r_k} \tau_s^{(k)}\nu_s^{(k)}. \tag{49}$$

Now a rational function of (z, v) which is adjoint for all finite values of the variable z must be an integral function of these variables.† Therefore a rational function of (z, v) possessing for finite values of the variable z the orders of coincidence given in (44) must be a polynomial in (z, v). Also a polynomial in (z, v) possessing for the value $z = \infty$ the orders of coincidence given in (45), will be included in the form given in (43) when j has been chosen large enough.‡ The polynomial $\psi(z, v)$ here in question, then, may be assumed to be the most general rational function whose orders of coincidence for finite values of the variable $z = a_k$ do not fall short of the numbers given in (44), while its orders of coincidence for the value $z = \infty$ are not less than the numbers given in (45).

On employing for brevity the notation $\bar{\tau}_1^{(k)}, \bar{\tau}_2^{(k)}, \ldots, \bar{\tau}_{r_k}^{(k)}$ to designate the numbers in (44) and the notation $\bar{\tau}_1^{(\infty)}, \bar{\tau}_2^{(\infty)}, \ldots, \bar{\tau}_{r_\infty}^{(\infty)}$ to designate the numbers in (45), we see that $\psi(z, v)$ is the most general rational function of (z, v) built on the basis $(\bar{\tau})$, while $H(z, v)$ is the most general rational function built on the basis (τ). Here the bases (τ) and $(\bar{\tau})$ are complementary to each other with regard to the level furnished by the function $F'_v(z, v)$, for we have

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* See *A. F.*, pp. 128, 129. † See *A. F.*, V, p. 84. ‡ See *A. F.*, pp. 126, 127.

$$\tau_s^{(k)} + \bar{\tau}_s^{(k)} = \mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}} \qquad (s = 1, 2, \ldots, r_k) \tag{50}$$

for all finite values of the variable $z = a_k$, and for the value $z = \infty$ we have

$$\tau_s^{(\infty)} + \bar{\tau}_s^{(\infty)} = \mu_s^{(\infty)} + 1 + \frac{1}{\nu_s^{(\infty)}} \qquad (s = 1, 2, \ldots, r_\infty). \tag{51}$$

The only limitation which we have here placed on the basis (τ) is that none of its numbers corresponding to any finite value of the variable z shall be positive. This limitation, however, is readily removed, as has been done in the text.* Assume, namely, that (τ) is any arbitrary basis of coincidences and $(\bar{\tau})$ the complementary basis whose coincidences satisfy the relations (50) and (51), and designate by $H(z, v)$ and $\bar{H}(z, v)$ respectively the most general rational functions built on these bases. Where $P(z)$ is any specific polynomial in z, the functions $\dfrac{H(z, v)}{P(z)}$ and $P(z)\,\bar{H}(z, v)$ are evidently also the most general rational functions built on certain complementary bases. Also by proper choice of $P(z)$ the basis for the former function will include no positive numbers corresponding to finite values of the variable z, and the complementary bases here in question will then come under the case for which we have already proved the complementary theorem. Furthermore the numbers of arbitrary constants involved in the expression of the functions $H(z, v)$ and $\bar{H}(z, v)$ are evidently the same as the numbers of arbitrary constants involved in the expressions of the functions $\dfrac{H(z, v)}{P(z)}$ and $P(z)\,\bar{H}(z, v) = \psi(z, v)$, while the sum of the orders of coincidence of the function $P(z)$ for all values of the variable z and for all the cycles corresponding is 0, and we therefore immediately derive

$$N_H = N_{\bar{H}} + n + \tfrac{1}{2} \sum_k \sum_{s=1}^{r_k} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}\right) \nu_s^{(k)} - \sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)}, \tag{52}$$

where (τ) and $(\bar{\tau})$ are any complementary bases whose coincidences satisfy the relations (50) and (51). By combination of (52) with the relations (50) and (51) we can readily write our result in the form

$$N_H + \tfrac{1}{2} \sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)} = N_{\bar{H}} + \tfrac{1}{2} \sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)} \nu_s^{(k)},$$

and the proof is immediate that this formula holds also when the relations (50) and (51) are replaced by the more general relations given in (2) and (3).

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* *A. F.*, pp. 134, 135.

[illegible] velve Surfaces of Darboux and the Transformation of Moutard.

By Luther Pfahler Eisenhart.

[illegible] wn fact that the problem of the infinitesimal deformation of [illegible] ent to the determination of the surfaces S_1, each of which [illegible] ondence with S in such a way that corresponding linear [illegible] al. The problem is equivalent also to the determination [illegible] h of which corresponds to S in such a way that the tangent [illegible] ding points are parallel and the asymptotic lines on either [illegible] nd to a conjugate system on the other. Darboux in the fourth [illegible] is *Leçons*[*] observes that the relation between S and S_1 being [illegible] cal, the surface S determines an infinitesimal deformation of S_1 and there [illegible] a surface S_3 associated thereby with S_1 which is analogous to the corresponding surface S_0 associated with S. In like manner the deformation of S_0 determined by S introduces a surface S_2 which corresponds to S_0 with orthogonality of linear elements. This process can be continued in both ways, and the coordinates of each new surface can be found directly. Darboux has derived the expressions for these coordinates, and has observed that the system is closed, consisting of twelve surfaces.

In the present paper we derive the same results in a similar manner, but we are concerned primarily with the determination of the reason for the closure. One of the fundamental reasons is expressed in Theorem III and the other is that the transformation by reciprocal polars with respect to the imaginary sphere, $x^2 + y^2 + z^2 + 1 = 0$, transforms a W-congruence into a congruence of the same kind. These results appear in §§ 1–4, and in § 5 it is shown that the twelve surfaces are distinct.

In § 6 we recall a theorem of Bianchi, namely that if S is a focal sheet of each of two W-congruences for which the other focal sheets are denoted by S_5

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[*] pp. 48–72.

and S_{52} respectively, each of the latter surfaces is a focal sheet of another congruence, which two congruences are such that the other focal sheets are one and the same surface S'. With each of the pairs of surfaces (S, S_5), (S, S_{52}), (S', S_5), (S', S_{52}) there may be associated ten other surfaces after the manner of § 3, giving in all forty surfaces. We consider in particular the twenty surfaces c[illegible] of the set S_0, S, S_1, S_3, S_5, S_7 and the analogous surfaces for each [illegible] above pairs. In § 7 it is shown that these twenty surfaces are dis[illegible] that S_0 and the similar surface S_{02} of the pair (S, S_{52}) are not [illegible] provided that the conjugate system on S_0 corresponding to t[illegible] on S does not have equal point invariants. This exceptional [illegible] in §§ 8–10, and it is found incidentally that the problem in[illegible] the determination of isothermal-conjugate systems of lin[illegible]

1. Let ξ, η, ζ be three particular solutions of an e[illegible]

$$\frac{\partial^2 \theta}{\partial u\, \partial v} = M\theta,$$

[illegible]

where M is a function of u and v. Upon the surface S whos[illegible] are defined by quadratures of the form

$$\frac{\partial x}{\partial u} = -\begin{vmatrix} \eta & \zeta \\ \frac{\partial \eta}{\partial u} & \frac{\partial \zeta}{\partial u} \end{vmatrix}, \qquad \frac{\partial x}{\partial v} = \begin{vmatrix} \eta & \zeta \\ \frac{\partial \eta}{\partial v} & \frac{\partial \zeta}{\partial v} \end{vmatrix}$$

the parametric curves are the asymptotic lines.*

Let σ be any particular solution of equation (1). This equation may accordingly be written

$$\frac{\partial^2 \theta}{\partial u\, \partial v} = \frac{1}{\sigma} \frac{\partial^2 \sigma}{\partial u\, \partial v}\, \theta. \tag{3}$$

The surface S_1, whose coordinates x_1, y_1, z_1 are given by the quadratures

$$\frac{\partial x_1}{\partial u} = -\sigma^2 \frac{\partial}{\partial u}\left(\frac{\xi}{\sigma}\right), \qquad \frac{\partial x_1}{\partial v} = \sigma^2 \frac{\partial}{\partial v}\left(\frac{\xi}{\sigma}\right), \tag{4}$$

and similar equations in y_1 and z_1, corresponds to S with orthogonality of linear elements; that is,

$$dx\, dx_1 + dy\, dy_1 + dz\, dz_1 = 0. \tag{5}$$

Moreover, the parametric curves on S_1 form a conjugate system with equal point invariants. In fact, x_1, y_1, z_1 are solutions of the equation

$$\frac{\partial^2 \phi}{\partial u\, \partial v} = \frac{\partial \log \sigma}{\partial v} \frac{\partial \phi}{\partial u} + \frac{\partial \log \sigma}{\partial u} \frac{\partial \phi}{\partial v}. \tag{6}$$

For the sake of brevity we shall say that S_1 is an *ortho-surface* of S.

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* Bianchi, *Lezioni*, Vol. I, p. 163; German translation, p. 132; also, Eisenhart, *Differential Geometry*, p. 198.

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5) may be replaced by the three equations

$$\left.\begin{aligned} dx_1 &= y_0\,dz - z_0\,dy, \\ dy_1 &= z_0\,dx - x_0\,dz, \\ dz_1 &= x_0\,dy - y_0\,dx, \end{aligned}\right\} \tag{7}$$

ce S_0 whose coordinates are x_0, y_0, z_0 is the surface associate tesimal deformation of the latter determined by S_1. We shall elation between the three surfaces S, S_1, S_0 by saying that S_0 *of S corresponding to S_1*, and that S_1 is *the ortho-surface of S* and that *to S_0*.

correspond with parallelism of tangent planes, the parametric S_0 form a conjugate system with equal tangential invariants.

When the values from (2) and (4) are substituted in equations (7), we obtain

$$\left.\begin{aligned} x_0 F(\xi, \eta, \zeta) = -F(\sigma, \eta, \zeta), \quad y_0 F(\xi, \eta, \zeta) = -F(\xi, \sigma, \zeta), \\ z_0 F(\xi, \eta, \zeta) = -F(\xi, \eta, \sigma), \end{aligned}\right\} \tag{8}$$

where the function $F(\theta_1, \theta_2, \theta_3)$ is defined by

$$F(\theta_1, \theta_2, \theta_3) = \begin{vmatrix} \theta_1 & \theta_2 & \theta_3 \\ \frac{\partial \theta_1}{\partial u} & \frac{\partial \theta_2}{\partial u} & \frac{\partial \theta_3}{\partial u} \\ \frac{\partial \theta_1}{\partial v} & \frac{\partial \theta_2}{\partial v} & \frac{\partial \theta_3}{\partial v} \end{vmatrix}. \tag{9}$$

Since the relation between S and S_1 is perfectly reciprocal, there exists a unique surface S_3 which is associate to S_1 corresponding to S. Its coordinates are given by equations of the form

$$dx = y_3\,dz_1 - z_3\,dy_1. \tag{10}$$

By means of (2) and (4) we find that

$$x_3 = \frac{\xi}{\sigma}, \quad y_3 = \frac{\eta}{\sigma}, \quad z_3 = \frac{\zeta}{\sigma}. \tag{11}$$

One consequence of this result is the theorem:

THEOREM I. *The radius vector of an associate surface of a surface S is parallel to the normal to the corresponding ortho-surface.*

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* *American Journal*, Vol. XXIV (1902), p. 186; *Differential Geometry*, p. 382.

In consequence of the fundamental identity

$$\sigma F(\xi, \eta, \zeta) = \xi F(\sigma, \eta, \zeta) + \eta F(\xi, \sigma, \zeta) +$$

from (8) and (11) follows the relation

$$x_0 x_3 + y_0 y_3 + z_0 z_3 + 1 = 0.$$

From (7) we have also

$$\Sigma x_0 \frac{\partial x_1}{\partial u} = 0, \quad \Sigma x_0 \frac{\partial x_1}{\partial v} = 0,$$

which in consequence of (4) and (11) may be written

$$\Sigma x_0 \frac{\partial x_3}{\partial u} = 0, \quad \Sigma x_0 \frac{\partial x_3}{\partial v} = 0.$$

Hence we have

THEOREM II. *The surfaces S_0 and S_3 are polar reciprocals of one another with respect to the imaginary sphere*

$$x^2 + y^2 + z^2 + 1 = 0. \tag{15}$$

2. Since the relation between associate surfaces is reciprocal, there is a surface S_5 which is the ortho-surface of S_3 corresponding to S_1. Its coordinates x_5, y_5, z_5 are given by equations of the form

$$dx_5 = y_1 dz_3 - z_1 dy_3. \tag{16}$$

If these equations be subtracted from the corresponding equations (10), and the resulting equations be integrated, we have

$$\left.\begin{aligned} x_5 - x &= y_1 z_3 - z_1 y_3, \\ y_5 - y &= z_1 x_3 - x_1 z_3, \\ z_5 - z &= x_1 y_3 - y_1 x_3; \end{aligned}\right\} \tag{17}$$

in this case we have taken the additive constants equal to zero, thus assuming a particular position in space for S_5.

If we put

$$\xi_1 = \frac{x_1}{\sigma}, \quad \eta_1 = \frac{y_1}{\sigma}, \quad \zeta_1 = \frac{z_1}{\sigma}, \tag{18}$$

then by THEOREM I the quantities ξ_1, η_1, ζ_1 are proportional to the direction-cosines of the normal to S_5.

By means of (11) and (18) equations (17) are reducible to

$$\left.\begin{aligned} x_5 - x &= \eta_1 \zeta - \zeta_1 \eta, \\ y_5 - y &= \zeta_1 \xi - \xi_1 \zeta, \\ z_5 - z &= \xi_1 \eta - \eta_1 \xi, \end{aligned}\right\} \tag{19}$$

and equations (4) become

$$\frac{\partial}{\partial u}\left(\sigma\xi_1\right) = -\sigma^2\frac{\partial}{\partial u}\left(\frac{\xi}{\sigma}\right), \quad \frac{\partial}{\partial v}\left(\sigma\xi_1\right) = \sigma^2\frac{\partial}{\partial v}\left(\frac{\xi}{\sigma}\right). \tag{20}$$

From these equations we find that ξ_1, η_1, ζ_1 are solutions of the equation

$$\frac{\partial^2\theta_1}{\partial u\,\partial v} = \sigma\frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{\sigma}\right).\theta. \tag{21}$$

Moreover, from (19) it follows that

$$\frac{\partial x_5}{\partial u} = -\begin{vmatrix} \eta_1 & \zeta_1 \\ \frac{\partial\eta_1}{\partial u} & \frac{\partial\zeta_1}{\partial u}\end{vmatrix}, \quad \frac{\partial x_5}{\partial v} = \begin{vmatrix} \eta_1 & \zeta_1 \\ \frac{\partial\eta_1}{\partial v} & \frac{\partial\zeta_1}{\partial v}\end{vmatrix}. \tag{22}$$

Hence the parametric curves on S_5 are its asymptotic lines.

From (19) it is seen that

$$\Sigma\xi(x_5 - x) = 0, \quad \Sigma\xi_1(x_5 - x) = 0.$$

Hence the line joining corresponding points on S and S_5 is tangent to both surfaces, and as the asymptotic lines on these surfaces correspond these joins form a *W*-congruence.

For convenience we represent the relations between the surfaces S, S_0, S_1, S_3, S_5 as in Fig. 1, where a vertical join indicates that the joined surfaces are associate to one another, and a horizontal line joins a surface and an ortho-surface.

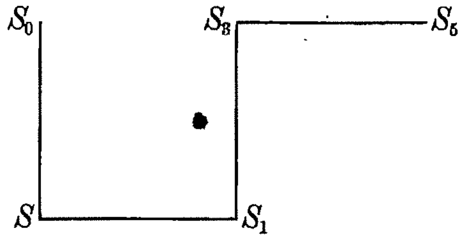

FIG. 1.

We shall prove the following converse result:

THEOREM III. *Let S and S_5 be the focal sheets of a W-congruence; if S_1 is an ortho-surface of S and S_3 is the corresponding associate surface of S_1, and also if S_3 is an ortho-surface of S_5, then S_1 is the associate of S_3 corresponding to S_5.*

Bianchi has shown* that if S and S_5 are the focal sheets of a *W*-congruence referred to the asymptotic lines of these surfaces, they may be defined by equations of the form (2) and (22) with the relations (19) and (20) between the quantities involved.

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* Vol. II, pp. 51–56; German translation, pp. 316–319; also, Eisenhart, *Differential Geometry*, pp. 417–420.

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Since S_3 is an ortho-surface of S_5, it follows from (4) that

$$\frac{\partial x_3}{\partial u} = -\sigma_1^2 \frac{\partial}{\partial u}\left(\frac{\xi_1}{\sigma_1}\right), \quad \frac{\partial x_3}{\partial v} = \sigma_1^2 \frac{\partial}{\partial v}\left(\frac{\xi_1}{\sigma_1}\right), \tag{23}$$

where σ_1 is a solution of equation (21). But from the definition of S_3, its coordinates are of the form (11). When this value is substituted in the first of equations (23), we obtain

$$\left(\frac{1}{\sigma} - \sigma_1\right)\frac{\partial \xi_1}{\partial u} + \left(\frac{1}{\sigma^2}\frac{\partial \sigma}{\partial u} + \frac{\partial \sigma_1}{\partial u}\right)\xi_1 = 0.$$

Since this equation holds also for η_1 and ζ_1, we must have $\sigma_1 = 1/\sigma$, and consequently the theorem is proved.

3. The process by which we obtained the surfaces S_3 and S_5 may be continued. Thus there is a surface S_7 which is the associate to S_5 corresponding to S_3. In turn we obtain surfaces S_9 and S_{11}, represented in Fig. 2, where the lines have the same significance as in Fig. 1.

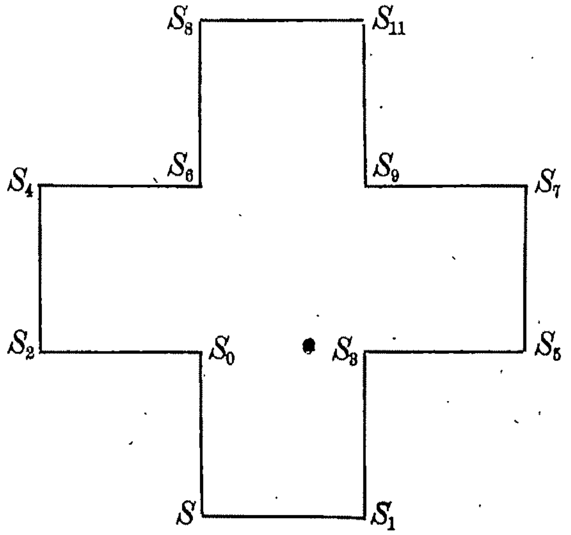

FIG. 2.

In order to obtain the coordinates x_7, y_7, z_7 of S_7, we observe that by THEOREM I the radius vector of S_7 is parallel to the normal to S_3, consequently to the normal to S_1, and therefore it is parallel to the radius vector of S_0. Combining this result with the fact that S_7 and S_1 are polar reciprocal with respect to the sphere (15), we have

$$\frac{x_7}{x_0} = \frac{y_7}{y_0} = \frac{z_7}{z_0} = -\frac{1}{\Sigma x_0 x_1}. \tag{24}$$

By reasoning of the same sort, we obtain by means of (18) the coordinates of S_{11}. They are given by

$$\frac{x_{11}}{\xi_1} = \frac{y_{11}}{\eta_1} = \frac{z_{11}}{\zeta_1} = -\frac{1}{\Sigma\,\xi_1 x_5} = -\frac{1}{\omega_1}, \tag{25}$$

where

$$\omega_1 = \Sigma\,\xi_1 x. \tag{26}$$

Since S_8 and S_9 are the focal sheets of a *W*-congruence, we have equations analogous to (17), namely

$$x_9 - x_8 = y_5 z_7 - z_5 y_7,$$

and similar expressions for y_9 and z_9. In consequence of (11), (19), (24), the above equation is reducible to

$$x_9 = \frac{\xi}{\sigma} - \frac{(y + \zeta_1 \xi - \zeta \xi_1) z_0 - (z + \xi_1 \eta - \xi \eta_1) y_0}{\Sigma\, x_0 x_1}. \tag{27}$$

4. In a similar manner we may proceed from S in the other direction and obtain a suite of surfaces S_0, S_2, ..., represented in Fig. 2. It is our purpose to show that S_8 and S_{11} are ortho-surfaces of one another corresponding to S_6 and S_9, and consequently that there is a closed cycle of twelve surfaces, and thus to justify the horizontal join of S_8 and S_{11} in Fig. 2.

The surfaces S_1 and S_2 are the focal sheets of a *W*-congruence, and consequently we have

$$x_2 = x_1 + y z_0 - z y_0, \tag{28}$$

and similar expressions for y_2 and z_2.

It can likewise be shown that the coordinates of S_8 are given by

$$\frac{x_8}{x} = \frac{y_8}{y} = \frac{z_8}{z} = -\frac{1}{\Sigma x x_1} = -\frac{1}{\sigma\omega_1}. \tag{29}$$

The surfaces S_8 and S_7 are polar reciprocals of S_2 and S_1 respectively with respect to the sphere (15). Since the relation of polar reciprocals with respect to this sphere is a contact transformation which preserves asymptotic lines, the surfaces S_8 and S_7 are the focal sheets of a *W*-congruence. Hence the coordinates of these surfaces are related by equations analogous to (17). These equations are in fact

$$x_8 - x_7 = y_9 z_{11} - z_9 y_{11}, \tag{30}$$

and similarly for y and z; for, the equation (30) is satisfied identically by the values from (24), (25), (27) and (29). Hence S_8 is the ortho-surface of S_{11} corresponding to S_9.

Since S_8 and S_{11} are ortho-surfaces, and since S_4 and S_{11} are the focal sheets of a W-congruence, being the polar reciprocals of S and S_5 respectively, it follows from THEOREM III that S_{11} is the ortho-surface of S_8 corresponding to S_6. Hence we have the following theorem of Darboux:

THEOREM IV. *Let S be any surface, S_1 an ortho-surface, and S_0 the corresponding associate surface; nine other surfaces can be found directly which form with these three a closed cycle of twelve surfaces, such that of any three contiguous surfaces of the cycle, S_a, S_b, S_c, either S_a is an ortho-surface of S_b and S_c the corresponding associate surface, or vice-versa.*

We shall determine whether the twelve surfaces are distinct. To this e remark that as the asymptotic lines are parametric on S, S_5, S_4 and S_{11}, oincides with any of the twelve surfaces it must be one of the latter three. e surface S cannot correspond with S_4 unless it be the sphere (15).[*] From (17) it follows that if S and S_5 coincide

$$\frac{x_8}{x_1} = \frac{y_8}{y_1} = \frac{z_8}{z_1}.$$

Since S_1 and S_8 are associate surfaces referred to their common conjugate system, we must[†] have

$$\frac{\partial x_8}{\partial u} = \lambda \frac{\partial x_1}{\partial u}, \quad \frac{\partial x_8}{\partial v} = -\lambda \frac{\partial x_1}{\partial v},$$

and similar equations in y and z, where λ is in general a function of u and v. Since these sets of equations are evidently inconsistent, the surfaces S and S_5 are always distinct.

It remains for us to consider the possible coincidence of S and S_{11}. From (25) and (18) it follows that if S and S_{11} coincide, we have

$$x_1 = tx, \quad y_1 = ty, \quad z_1 = tz,$$

where t is a factor of proportionality. Suppose that S and S_1 are referred to the system of lines which is orthogonal for both surfaces. In this case

$$\Sigma \frac{\partial x_1}{\partial u}\frac{\partial x_1}{\partial v} = 0, \quad \Sigma \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} = 0.$$

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[*] It should be remarked that if S is the unit sphere, S_4 is the same sphere, but diametrically opposite points correspond (cf. § 10).

[†] Eisenhart, *Differential Geometry*, p. 380.

If we substitute the above values in the first of these equations and reduce the result by means of the second, we obtain

$$t\frac{\partial t}{\partial u}\Sigma x\frac{\partial x}{\partial v}+t\frac{\partial t}{\partial v}\Sigma x\frac{\partial x}{\partial u}+\frac{\partial t}{\partial u}\frac{\partial t}{\partial v}\Sigma x^2=0.$$

From (5) it follows that

$$\Sigma\frac{\partial x}{\partial u}\frac{\partial x_1}{\partial v}+\Sigma\frac{\partial x}{\partial v}\frac{\partial x_1}{\partial u}=0.$$

When the above values of x_1, y_1, z_1 are substituted in this equation, we obtain

$$\frac{\partial t}{\partial v}\Sigma x\frac{\partial x}{\partial u}+\frac{\partial t}{\partial u}\Sigma x\frac{\partial x}{\partial v}=0.$$

Comparing these two equations for the determination of t, we see that t must be a constant. But in this case equation (5) reduces to $\Sigma dx^2=0$, which evidently is impossible. Hence S does not coincide with any other of the twelve surfaces. Moreover, since any surface of the cycle can be taken as the initial surface, it follows that all of the twelve surfaces are distinct.

6. Bianchi* has shown that if σ_1 and σ_2 are particular solutions of equation (1), the functions σ_1' and σ_2', given by

$$\frac{\partial}{\partial u}\left(\sigma_1\sigma_1'\right)=-\sigma_1^2\frac{\partial}{\partial u}\left(\frac{\sigma_2}{\sigma_1}\right),\quad \frac{\partial}{\partial v}\left(\sigma_1\sigma_1'\right)=\sigma_1^2\frac{\partial}{\partial v}\left(\frac{\sigma_2}{\sigma_1}\right),\tag{31}$$

$$\frac{\partial}{\partial u}\left(\sigma_2\sigma_2'\right)=-\sigma_2^2\frac{\partial}{\partial u}\left(\frac{\sigma_1}{\sigma_2}\right),\quad \frac{\partial}{\partial v}\left(\sigma_2\sigma_2'\right)=\sigma_2^2\frac{\partial}{\partial v}\left(\frac{\sigma_1}{\sigma_2}\right),\tag{32}$$

satisfy the conditions

$$\sigma_2\sigma_2'=-\sigma_1\sigma_1',\quad \sigma_1'\frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{\sigma_1'}\right)=\sigma_2'\frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{\sigma_2'}\right),\tag{33}$$

and are solutions of the respective equations

$$\frac{\partial^2\theta_1}{\partial u\,\partial v}=\sigma_1\frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{\sigma_1}\right)\theta_1,\tag{34}$$

$$\frac{\partial^2\theta_2}{\partial u\,\partial v}=\sigma_2\frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{\sigma_2}\right)\theta_2.\tag{35}$$

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* Vol. II, p. 70.

Let ξ_1, η_1, ζ_1; ξ_2, η_2, ζ_2 be the solutions of equations (34) and (35) respectively, given by quadratures of the form (20). It is readily shown that the functions ξ', η', ζ', defined by

$$\xi' = \xi + t(\xi_1 - \xi_2), \quad \eta' = \eta + t(\eta_1 - \eta_2), \quad \zeta' = \zeta + t(\zeta_1 - \zeta_2), \tag{36}$$

where

$$t = -\frac{\sigma_2}{\sigma_1'} = \frac{\sigma_1}{\sigma_2'}, \tag{37}$$

are solutions of the equation

$$\frac{\partial^2 \theta'}{\partial u\, \partial v} = \sigma_1' \frac{\partial^2}{\partial u\, \partial v}\left(\frac{1}{\sigma_1'}\right)\theta' = \sigma_2' \frac{\partial^2}{\partial u\, \partial v}\left(\frac{1}{\sigma_2'}\right)\theta'. \tag{38}$$

The functions ξ_2, η_2, ζ_2 determine a surface S_{52} whose coordinates are given by equations analogous to (22). The surfaces S and S_{52} are the focal sheets of a W-congruence, and their coordinates are in the relation

$$x_{52} = x + (\eta_2 \zeta - \zeta_2 \eta), \quad y_{52} = y + (\zeta_2 \xi - \xi_2 \zeta), \quad z_{52} = z + (\xi_2 \eta - \eta_2 \xi). \tag{39}$$

Furthermore, the surface S', defined by

$$\left.\begin{aligned} x' &= x + t(\eta_1 \zeta_2 - \zeta_1 \eta_2), \\ y' &= y + t(\zeta_1 \xi_2 - \xi_1 \zeta_2), \\ z' &= z + t(\xi_1 \eta_2 - \eta_1 \xi_2), \end{aligned}\right\} \tag{40}$$

is such that S_5 and S' are focal sheets of a W-congruence, and likewise S_{52} and S'. For convenience we shall say that the four surfaces S, S_5, S_{52}, S' form a *quatern*. The asymptotic lines correspond on all four surfaces, and the latter are determined by quadratures when two integrals of equation (1) are known.

With each of the three W-congruences thus introduced there are associated four surfaces analogous to S_0, S_1, S_3, S_7. They are denoted as follows:

$$\left.\begin{array}{llllll} S_0, & S, & S_1, & S_3, & S_5, & S_7; \\ S_{02}, & S, & S_{12}, & S_{32}, & S_{52}, & S_{72}; \\ S_0', & S', & S_1'', & S_3', & S_5, & S_7''; \\ S_{02}', & S', & S_{12}', & S_{32}', & S_{52}, & S_{72}''. \end{array}\right\} \tag{41}$$

It is desirable that we should know under what conditions all of these surfaces are distinct.

7. Since the parametric curves on the surfaces of the second and fifth columns of (41) are the asymptotic lines on these surfaces, and since this is true of none of the other surfaces, it follows that these four surfaces are distinct from all the others.

In a similar manner, the surfaces of the first and last columns are characterized by the fact that the parametric curves form a conjugate system with equal tangential invariants, and in general the surfaces of the third and fourth columns do not possess this property. But two of these surfaces cannot coincide, unless the same is true of two of the surfaces S, S_5, S_{52}, S'.

From (11) and (18) it follows that the coordinates of the surfaces of the third and fourth columns are of the form

$$\left.\begin{aligned} x_1 &= \xi_1\sigma_1, & x_{12} &= \xi_2\sigma_2, & x'_3 &= \xi'\sigma'_1, & x'_{32} &= \xi'\sigma'_2, \\ x_3 &= \frac{\xi}{\sigma_1}, & x_{32} &= \frac{\xi}{\sigma_2}, & x'_1 &= \frac{\xi_1}{\sigma'_1}, & x'_{12} &= \frac{\xi_2}{\sigma'_2}. \end{aligned}\right\} \tag{42}$$

The coincidence of S_1 with any of these surfaces other than S'_1 necessitates the coincidence of two of the surfaces S, S_5, S_{52}, S'.

We consider the case where S_1 and S'_1 are coincident. From (42) it follows that we must have $\sigma_1\sigma'_1 = 1$. Now equations (31), (32) reduce to

$$\sigma_2 = \varkappa\sigma_1, \tag{43}$$

where $\varkappa$ denotes a constant. Hence equations (34) and (35) are equivalent. From (20) we find that

$$\frac{\partial}{\partial u}\left(\sigma_1\xi_2\right) = \frac{\partial}{\partial u}\left(\sigma_1\xi_1\right), \quad \frac{\partial}{\partial v}\left(\sigma_1\xi_2\right) = \frac{\partial}{\partial v}\left(\sigma_1\xi_1\right),$$

whence we have

$$\xi_2 = \xi_1 + \frac{a}{\sigma_1}, \quad \eta_2 = \eta_1 + \frac{b}{\sigma_1}, \quad \zeta_2 = \zeta_1 + \frac{c}{\sigma_1}, \tag{44}$$

a, b, c being constants.

Since $\sigma_1\sigma'_1 = 1$, equation (38) is equivalent to (1), and equations (36) reduce to

$$\xi' = \xi + \varkappa a\sigma_1, \quad \eta' = \eta + \varkappa b\sigma_1, \quad \zeta' = \zeta + \varkappa c\sigma_1. \tag{45}$$

From (8) and analogous formulas we find that S_7 and S'_7 coincide, that the surfaces of each pair S_0, S_{02}; S_{72}, S'_{72}; S'_0, S'_{02} are homothetic transforms of one another, and that

$$\frac{x_{72}}{x_7} = \frac{y_{72}}{y_7} = \frac{z_{72}}{z_7} = \frac{1}{\varkappa\left(1 - ax_7 - by_7 - cz_7\right)},$$

$$\frac{x'_0}{x_0} = \frac{y'_0}{y_0} = \frac{z'_0}{z_0} = \frac{1}{1 - \varkappa\left(ax_0 + by_0 + cz_0\right)}.$$

By means of (42), (43), (44), (45) it can be shown that the surfaces S_{12}, S'_{12} can be obtained from S_1 by homothetic transformations and translations; and similarly S_{32}, S'_3, S'_{32} from S_3.

The foregoing results may be stated thus:

THEOREM V. *When σ_1 and σ_2 are not in constant ratio, the twenty surfaces* (41) *are distinct, provided that the parametric curves on any of the surfaces in the first or last column do not form a conjugate system with equal point invariants.*

8. Thus far we have excluded the case where the parametric conjugate system on S_0 has equal point invariants as well as equal tangential invariants; that is, where

$$\frac{\partial}{\partial u}\begin{Bmatrix}12\\1\end{Bmatrix}_0 = \frac{\partial}{\partial v}\begin{Bmatrix}12\\2\end{Bmatrix}_0, \tag{46}$$

the symbols $\begin{Bmatrix}rs\\t\end{Bmatrix}_0$ being formed with respect to the linear element of S_0. We consider this case.

The Codazzi equations* for S_0 are of the form

$$\left.\begin{aligned}\frac{\partial \log D_0}{\partial v} &= \begin{Bmatrix}12\\1\end{Bmatrix}' - \frac{D''_0}{D_0}\begin{Bmatrix}11\\2\end{Bmatrix}',\\ \frac{\partial \log D''_0}{\partial u} &= \begin{Bmatrix}12\\2\end{Bmatrix}' - \frac{D_0}{D''_0}\begin{Bmatrix}22\\1\end{Bmatrix}',\end{aligned}\right\} \tag{47}$$

the symbols $\begin{Bmatrix}rs\\t\end{Bmatrix}'$ being formed with respect to the linear element of the spherical representation of S and S_0, namely

$$d\sigma^2 = \mathcal{E}du^2 + 2\mathcal{F}du\,dv + \mathcal{G}dv^2. \tag{48}$$

In consequence of the following relations:†

$$\frac{D''_0}{D_0}\begin{Bmatrix}11\\2\end{Bmatrix}' = -\begin{Bmatrix}12\\1\end{Bmatrix}_0, \quad \frac{D_0}{D''_0}\begin{Bmatrix}22\\1\end{Bmatrix}' = -\begin{Bmatrix}12\\2\end{Bmatrix}_0, \tag{49}$$

equations (47) are reducible to

$$\left.\begin{aligned}\frac{\partial \log D_0}{\partial v} &= \begin{Bmatrix}12\\1\end{Bmatrix}' + \begin{Bmatrix}12\\1\end{Bmatrix}_0,\\ \frac{\partial \log D''_0}{\partial u} &= \begin{Bmatrix}12\\2\end{Bmatrix}' + \begin{Bmatrix}12\\2\end{Bmatrix}_0.\end{aligned}\right\} \tag{47'}$$

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* Bianchi, I, 166; German translation, p. 134; Eisenhart, p. 161.

† Bianchi, I, 167; German translation, p. 135; Eisenhart, p. 201.

In consequence of (46) and the fact that the asymptotic lines on S are parametric, we have from (47′)

$$\frac{\partial^2}{\partial u\,\partial v}\log\frac{D_0}{D_0''}=0.$$

Hence the parametric system on S_0 is isothermal-conjugate. As the parameters of the asymptotic lines on S were not of any particular kind, the parameters may be chosen so that $D_0=\varepsilon D_0''$, where ε is ± 1 according as the curvature of S_0 is positive or negative.

In this case equations (47) are reducible to

$$\left.\begin{aligned}\frac{\partial\log D_0}{\partial v}&=\left\{{12\atop 1}\right\}'-\varepsilon\left\{{11\atop 2}\right\}',\\ \frac{\partial\log D_0}{\partial u}&=\left\{{12\atop 2}\right\}'-\varepsilon\left\{{22\atop 1}\right\}'.\end{aligned}\right\}\tag{50}$$

In order that these equations be consistent, we must have

$$\frac{\partial}{\partial u}\left\{{11\atop 2}\right\}'=\frac{\partial}{\partial v}\left\{{22\atop 1}\right\}'.\tag{51}$$

When this condition is satisfied, two sets of solutions of (47) follow from (50) by quadratures. One set is such that $D_0=D_0''$, and the other that $D_0=-D_0''$. The further determination of the surfaces determined by these values requires quadratures of the form*

$$\left.\begin{aligned}\frac{\partial x_0}{\partial u}&=\frac{D_0}{\sqrt{\mathcal{E}\mathcal{G}-\mathcal{F}^2}}\left(-\mathcal{G}\frac{\partial X}{\partial u}+\mathcal{F}\frac{\partial X}{\partial v}\right),\\ \frac{\partial x_0}{\partial v}&=\frac{D_0''}{\sqrt{\mathcal{E}\mathcal{G}-\mathcal{F}^2}}\left(\mathcal{F}\frac{\partial X}{\partial v}-\mathcal{E}\frac{\partial X}{\partial u}\right).\end{aligned}\right\}\tag{52}$$

Hence we have the theorem:

THEOREM VI. *When the spherical representation of the asymptotic lines of a surface S satisfies the condition* (51), *one can find by quadratures two surfaces S_0 and S_{02}, associate to S, upon which the parametric lines form an isothermal-conjugate system, the parameters being such that*

$$D_0=D_0'',\qquad D_{02}=-D_{02}'';\tag{53}$$

and S_0 and S_{02} are associate surfaces of one another.

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* Bianchi, I, 166; German translation, p. 134; Eisenhart, p. 200.

The last statement follows from the fact that $D_0 D''_{02} + D''_0 D_{02} = 0$.*

9. In order to connect up this result with the results of the preceding sections, we remark that another form of equations (2) is:†

$$\left.\begin{aligned} \frac{\partial x}{\partial u} &= \frac{D'}{\sqrt{\mathcal{E}\mathcal{G}-\mathcal{F}^2}}\left(\mathcal{F}\frac{\partial X}{\partial u} - \mathcal{E}\frac{\partial X}{\partial v}\right), \\ \frac{\partial x}{\partial v} &= \frac{D'}{\sqrt{\mathcal{E}\mathcal{G}-\mathcal{F}^2}}\left(-\mathcal{G}\frac{\partial X}{\partial u} + \mathcal{F}\frac{\partial X}{\partial v}\right). \end{aligned}\right\} \tag{54}$$

From (52) and (54) it follows that

$$\frac{\partial x_0}{\partial u} = \frac{D_0}{D'}\frac{\partial x}{\partial v}, \quad \frac{\partial x_0}{\partial v} = \frac{D''_0}{D'}\frac{\partial x}{\partial u}. \tag{55}$$

From equations (8) and (2) we find that

$$\left.\begin{aligned} \frac{x_{0u}}{x_v} &= \frac{1}{F^2(\xi, \eta, \zeta)}\begin{vmatrix} \xi & & & \\ & \eta_u & & \\ & & \zeta_v & \\ & & & \sigma_{1uu} \end{vmatrix}, \\ \frac{x_{0v}}{x_u} &= \frac{1}{F^2(\xi, \eta, \zeta)}\begin{vmatrix} \xi & & & \\ & \eta_u & & \\ & & \zeta_v & \\ & & & \sigma_{1vv} \end{vmatrix}, \end{aligned}\right\} \tag{56}$$

where the subscripts u and v indicate differentiation with respect to these parameters, where σ of (8) has been replaced by σ_1, and where the omitted terms are similar to those given.

Hence a necessary and sufficient condition that S_0 be of the kind defined in THEOREM VI is that σ_1 satisfy the condition

$$\begin{vmatrix} \xi & & & \\ & \eta_u & & \\ & & \zeta_v & \\ & & & \sigma_{1uu} \end{vmatrix} = \begin{vmatrix} \xi & & & \\ & \eta_u & & \\ & & \zeta_v & \\ & & & \sigma_{1vv} \end{vmatrix}. \tag{57}$$

This equation may be written in the form

$$\begin{vmatrix} \xi_u\sigma_1 - \xi\sigma_{1u}, & \eta_u\sigma_1 - \eta\sigma_{1u}, & \zeta_u\sigma_1 - \zeta\sigma_{1u} \\ \xi_v\sigma_1 - \xi\sigma_{1v}, & \ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots \\ \xi_{uu}\sigma_1 - \xi\sigma_{1uu}, & \ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots \end{vmatrix} = \begin{vmatrix} \xi_u\sigma_1 - \xi\sigma_{1u}, & \ldots\ldots, & \ldots\ldots \\ \xi_v\sigma_1 - \xi\sigma_{1v}, & \ldots\ldots, & \ldots\ldots \\ \xi_{vv}\sigma_1 - \xi\sigma_{1vv}, & \ldots\ldots, & \ldots\ldots \end{vmatrix} \tag{58}$$

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* In the *Mathematische Annalen*, Vol. LXII (1906), pp. 524–529, we have given examples of surfaces which satisfy the requirements of THEOREM VI.

† Bianchi, I, 158; Eisenhart, p. 192.

which by means of (20) is equivalent to

$$\begin{vmatrix} \xi_1 & & & \\ & \eta_{1u} & & \\ & & \zeta_{1v} & \\ & & & \left(\frac{1}{\sigma_1}\right)_{uu} \end{vmatrix} = -\begin{vmatrix} \xi_1 & & & \\ & \eta_{1u} & & \\ & & \zeta_{1v} & \\ & & & \left(\frac{1}{\sigma_1}\right)_{vv} \end{vmatrix}. \tag{59}$$

This equation results also from the following theorem which we have established elsewhere:*

THEOREM VII. *The asymptotic lines on any one of a set of three surfaces S, S_1, S_0 correspond to a conjugate system on each of the other two surfaces.* For, from this theorem it follows that

$$D_0 D_1'' + D_0'' D_1 = 0, \quad D_1 D_8'' + D_1'' D_8 = 0, \quad D_8 D_7'' + D_8'' D_7 = 0.$$

Since $D_0 = D_0''$, we find that $D_7 = -D_7''$, and consequently from equations analogous to (55) and (56) we obtain (59).

In like manner we find that the function σ_2 which determines the surface S_{02}, defined in THEOREM VI, must satisfy the condition

$$\begin{vmatrix} \xi & & & \\ & \eta_u & & \\ & & \zeta_v & \\ & & & \sigma_{2uu} \end{vmatrix} = -\begin{vmatrix} \xi & & & \\ & \eta_u & & \\ & & \zeta_v & \\ & & & \sigma_{2vv} \end{vmatrix}. \tag{60}$$

Since the parametric curves on S_7 form an isothermal-conjugate system, there is a surface Σ associate to S_6 which is also associate to S_7, by THEOREM VI. We inquire whether the surface Σ is the surface S_7' of (41). On this hypothesis the equation obtained by replacing ξ, η, ζ, σ_1 in (58) by ξ_1, η_1, ζ_1, σ_1' respectively must hold. By means of (20) and (31) this equation is reducible to

$$\begin{vmatrix} \xi\sigma_1' - \xi_1\sigma_2, & \ldots, & \ldots, & \sigma_1 \\ \xi_u\sigma_1' - \xi_1\sigma_{2u}, & \ldots, & \ldots, & \sigma_{1u} \\ \xi_v\sigma_1' - \xi_1\sigma_{2v}, & \ldots, & \ldots, & \sigma_{1v} \\ \xi_{uu}\sigma_1' - \xi_1\sigma_{2uu}, & \ldots, & \ldots, & \sigma_{1uu} \end{vmatrix} = -\begin{vmatrix} \xi\sigma_1' - \xi_1\sigma_2, & \ldots, & \ldots, & \sigma_1 \\ \xi_u\sigma_1' - \xi_1\sigma_{2u}, & \ldots, & \ldots, & \sigma_{1u} \\ \xi_v\sigma_1' - \xi_1\sigma_{2v}, & \ldots, & \ldots, & \sigma_{1v} \\ \xi_{vv}\sigma_1' - \xi_1\sigma_{2vv}, & \ldots, & \ldots, & \sigma_{1vv} \end{vmatrix} \tag{61}$$

If equations (57) and (60) be written in the abbreviated form

$$\left.\begin{aligned} A(\sigma_{1uu} - \sigma_{1vv}) + (B - B_1)\sigma_{1u} + (C - C_1)\sigma_{1v} + (D - D_1)\sigma_1 = 0, \\ A(\sigma_{2uu} + \sigma_{2vv}) + (B + B_1)\sigma_{2u} + (C + C_1)\sigma_{2v} + (D + D_1)\sigma_2 = 0; \end{aligned}\right\} \tag{62}$$

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* *Differential Geometry*, p. 384.

we remark that $A, B, \ldots, D_1$ are such that ξ, η, ζ satisfy the equations

$$\begin{aligned} A\,\xi_{uu} + B\,\xi_u + C\,\xi_v + D\,\xi &= 0, \\ A\,\xi_{vv} + B_1\,\xi_u + C_1\,\xi_v + D_1\,\xi &= 0. \end{aligned}$$

By means of these quantities equation (61) can be reduced to

$$\begin{vmatrix} \xi\,\sigma_1' - \xi_1\,\sigma_2, & & & \sigma_1 \\ \xi_u\,\sigma_1' - \xi_1\,\sigma_{2u}, & & & \sigma_{1u} \\ \xi_v\,\sigma_1' - \xi_1\,\sigma_{2v}, & & & \sigma_{1v} \\ 0, & 0, & 0, & \begin{array}{r} A(\sigma_{1uu} + \sigma_{1vv}) + (B + B_1)\,\sigma_{1u} \\ + (C + C_1)\,\sigma_{1v} + (D + D_1)\,\sigma_1 \end{array} \end{vmatrix} = 0. \qquad (63)$$

From (56) it follows that if both members of (57) vanish, S_0 is a point. Excluding this case, equation (63) reduces to

$$\begin{vmatrix} \xi_1 \cdots\cdots\cdots \sigma_1' \\ \xi \cdots\cdots\cdots \sigma_2 \\ \xi_u \cdots\cdots\cdots \sigma_{2u} \\ \xi_v \cdots\cdots\cdots \sigma_{2v} \end{vmatrix} = 0.$$

If we put

$$\xi_1 = a\,\xi + b\,\xi_u + c\,\xi_v, \qquad (64)$$

where a, b, c are functions of u and v, the first of equations (20) is reducible to

$$\xi\left[(a-1)\frac{\sigma_{1u}}{\sigma_1} + a_u + cM\right] + \xi_u\left(1 + a + \frac{\sigma_{1u}}{\sigma_1}\,b + b_u\right) + \xi_v\left(c\,\frac{\sigma_{1u}}{\sigma_1} + c_u\right) + b\,\xi_{uu} = 0. \qquad (65)$$

Since this equation is satisfied also by η, ζ and σ_2, it follows that unless the right-hand member of equation (60) is zero, all the coefficients in (65) are equal to zero. Similar results follow when the value (64) of ξ_1 is substituted in the second of (20). Combining these results we are led to the conclusion that S_{02} must be a point. Hence if equation (61) is to be satisfied, either S_0 or S_{02} must be a point. When S_0 is a point, both S_1 and S_8 are planes, and S_7 is a point.

In a similar manner it can be shown that the parameters of the parametric conjugate system on S'_{12} are isothermal-conjugate only when S_0 or S_{02} is a point.

In consequence of THEOREM VII the foregoing results may be stated thus:

THEOREM VIII. *If the conjugate parametric system on S_0 is such that $D_0 = D_0''$ and if S_{03} is a point, the conjugate parametric system on each of the surfaces* (41), *except S, S_5, S_{53}, S', is such that $D_i = D_i''$ or $D_i = -D_i''$; the pairs of surfaces S_0', S_{03}'; S_7, S_7' are associate surfaces; and this is the only case in which a group of surfaces* (41) *possesses these properties.*

10. We proceed to an investigation of the problem as to the coincidence of certain surfaces of the group (41), when S_0 and S_{03} satisfy the requirements of THEOREM VI, and neither is a point.

From the results of § 5 we know that S_0 cannot coincide with S_1, S_3, S_5 or S_7. Since conditions (53) are satisfied, it follows from THEOREM VII that

$$D_{13} = D_{13}'', \quad D_{33} = -D_{33}''.$$

Hence S_0 cannot coincide with S_{33}, but it remains for us to see whether S_0 and S_{13} can coincide.

On this hypothesis we have

$$\Sigma X \frac{\partial x_{13}}{\partial u} = 0, \quad \Sigma X \frac{\partial x_{13}}{\partial v} = 0. \tag{66}$$

It is known that in general*

$$\Sigma X \frac{\partial x_{13}}{\partial u} = \frac{D' \dfrac{\partial \phi}{\partial u} - D \dfrac{\partial \phi}{\partial v}}{K\sqrt{EG-F^2}}, \quad \Sigma X \frac{\partial x_{13}}{\partial v} = \frac{D'' \dfrac{\partial \phi}{\partial u} - D' \dfrac{\partial \phi}{\partial v}}{K\sqrt{EG-F^2}},$$

where ϕ is a solution of the equation

$$\frac{1}{\sqrt{EG-F^2}} \left\{ \frac{\partial}{\partial u}\left(\frac{D'' \dfrac{\partial \phi}{\partial u} - D' \dfrac{\partial \phi}{\partial v}}{K\sqrt{EG-F^2}} \right) + \frac{\partial}{\partial v}\left(\frac{D \dfrac{\partial \phi}{\partial v} - D' \dfrac{\partial \phi}{\partial u}}{K\sqrt{EG-F^2}} \right) \right\}$$
$$= \frac{2FD' - ED'' - GD}{EG-F^2}\phi. \tag{67}$$

In order that (66) be satisfied, ϕ must be a constant and consequently S must be a minimal surface, as is seen from (67). Hence an adjoint of a minimal surface may be taken for both S_0 and S_{13}.† In this case S_0 is a surface of negative curvature. Consequently in order that the conditions (53) may be preserved, we take the adjoint of S for S_{02} and S_1. When now we compare equations (7) with the formulas of Schwarz,* namely

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* Bianchi, II, 6; German translation, p. 290; Eisenhart, p. 375.

† Cf. *American Journal*, *loc. cit.*, p. 196.

$$\begin{aligned} dx_1 &= Ydz - Zdy, \\ dy_1 &= Zdx - Xdz, \\ dz_1 &= Xdy - Ydx, \end{aligned}$$

we notice that S_0 is the unit sphere. Since S is adjoint to S_1, we have

$$dx = -Ydz_1 + Zdy_1,$$

and similar expressions in y and z. Hence S_3 is the unit sphere also, but corresponding points of S_0 and S_3 are diametrically opposite. Since S_5 corresponds to the sphere S_3 with orthogonality of linear elements, S_5 is the mean surface of an isotropic congruence.† In consequence of THEOREM VIII neither S_0 nor S_{02} can coincide with any members of the group (41) other than those which we have considered.

We suppose now that S_0 is such that the first of equations (53) holds, but that the second of these equations is not satisfied by S_{02}. If S_0 and S_3'' were to coincide, S_7 and S_7'' would be such that $D_{7'} = -D_{7'}''$, $D_7 = -D_7''$, which is impossible. If S_0 and S_1' were to coincide, we should have $D_{7'} = D_{7'}''$, as well as $D_7 = -D_7''$, and consequently this is the same problem as determining whether S_7 and S_{32} can coincide, when both of the conditions (53) are satisfied. If S_7 and S_{32} coincide, so also do their polar reciprocals S_1 and S_{02}. But we have seen that this is true only when S is minimal, and S_1 and S_{02} are its adjoint.

It remains for us to determine under what conditions S_0 can coincide with S_{12}'' or S_{32}''. From (42) it follows that if S_0 and S_{12}'' coincide

$$\xi_2 = x_0 \sigma_2', \quad \eta_2 = y_0 \sigma_2', \quad \zeta_2 = z_0 \sigma_2'.$$

When these values are substituted in equation (35) and it is recalled that σ_2' also is a solution of this equation, we find that x_0, y_0, z_0 are solutions of

$$\frac{\partial^2 \theta}{\partial u\, \partial v} + \frac{\partial \log \sigma_2'}{\partial v} \frac{\partial \theta}{\partial u} + \frac{\partial \log \sigma_2'}{\partial u} \frac{\partial \theta}{\partial v} = 0.$$

By means of (49) the point equation of S_0 is reducible to

$$\frac{\partial^2 \theta}{\partial u\, \partial v} + \varepsilon \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' \frac{\partial \theta}{\partial u} + \varepsilon \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}' \frac{\partial \theta}{\partial v} = 0,$$

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* Bianchi, Vol. II, p. 337; German translation, p. 377; Eisenhart, p. 265.

† Darboux, IV, p. 95. In the subsequent pages Darboux determines all the surfaces of the group represented in Fig. 2, when S is a sphere.

where ε is ± 1 according as the curvature of S_0 is positive or negative. Hence σ_2' is given by the quadratures

$$\frac{\partial \log \sigma_2'}{\partial u} = \varepsilon \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}', \quad \frac{\partial \log \sigma_2'}{\partial v} = \varepsilon \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}'. \tag{68}$$

We have shown elsewhere* that when the condition (51) is satisfied, the surface S_0 is given by the quadratures

$$\frac{\partial x_0}{\partial u} = \mu \frac{\partial x}{\partial v}, \quad \frac{\partial x_0}{\partial v} = \varepsilon \mu \frac{\partial x}{\partial u},$$

where, according as ε is ± 1, S_0 is of positive or negative curvature, and μ is determined by

$$\frac{\partial \log \mu}{\partial v} = \varepsilon \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}, \quad \frac{\partial \log \mu}{\partial u} = \varepsilon \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}, \tag{69}$$

the Christoffel symbols being formed with respect to the linear element of S. If K denotes the total curvature of S and ρ is defined by

$$\frac{1}{\rho^2} = -K,$$

it is possible to derive from (68) and (69) the relation

$$\mu \sigma_2' = \rho^{\frac{3}{2}} \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2}, \tag{70}$$

where $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ have the same significance as in (48). Hence the determination of S_0 carries with it that of σ_2'. Likewise, when S_0 is known, the corresponding function σ_1 is found directly from (11), (13) and

$$\Sigma x_3 \frac{\partial x_0}{\partial u} = 0, \quad \Sigma x_3 \frac{\partial x_0}{\partial v} = 0.$$

In the next place σ_2 must be found, and from (32) it follows that this problem reduces to the quadratures

$$\frac{\partial \log \sigma_2}{\partial u} = -\frac{\dfrac{\partial \sigma_2'}{\partial u} + \dfrac{\partial \sigma_1}{\partial u}}{\sigma_2' - \sigma_1}, \quad \frac{\partial \log \sigma_2}{\partial v} = -\frac{\dfrac{\partial \sigma_2'}{\partial v} - \dfrac{\partial \sigma_1}{\partial v}}{\sigma_2' + \sigma_1}. \tag{71}$$

The condition of integrability of these equations is reducible to

$$\left(\frac{\partial^2 \sigma_2'}{\partial u \partial v} + M\sigma_2'\right)(\sigma_2' + \sigma_1)(\sigma_2' - \sigma_1) - 2\sigma_2'\left(\frac{\partial \sigma_2'}{\partial v} - \frac{\partial \sigma_1}{\partial v}\right)\left(\frac{\partial \sigma_2'}{\partial u} + \frac{\partial \sigma_1}{\partial u}\right) = 0.$$

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* *Mathematische Annalen*, Vol. LXII (1906), pp. 507, 520.

Moreover, it is readily shown that when this condition is satisfied, the function σ_2 given by (71) is a solution of equation (1). The solution of our problem depends therefore upon the determination of the surfaces S whose asymptotic lines satisfy the condition (51). As we have been unable to get the general solution of this problem,* we cannot determine the conditions under which S_0 and S'_{12} coincide. It should be remarked that from THEOREM VII it follows that if the conditions of the problem were satisfied the surfaces S_5, S_{52} and S' also would satisfy the condition (51).

The problem of the coincidence of S_0 and S'_{32} is similar to the preceding one, the difference being that equation (70) must be replaced by

$$\frac{\mu}{\sigma'_2} = \rho^{\frac{1}{4}} \sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2}.$$

Having thus considered the possibilities of S_0 conciding with other surfaces of the group (41), it is unnecessary to go any further. For, the only other type is S_1 and its polar reciprocal S_7 is of the same type as S_0.

PRINCETON, *April* 20, 1909.

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* *Mathematische Annalen*, Vol. LXII (1906), pp. 534–586. Here we have shown that minimal surfaces and pseudospherical surfaces of revolution satisfy the condition (51).

On the Problem of the Spherical Representation and the Characteristic Equations of Certain Classes of Surfaces.[*]

By Archer Everett Young.

§ 1. *Introduction.*

If systems of lines on the Gauss sphere cutting at right angles are given, the determination of the most general class of surfaces having these as the spherical representation of their lines of curvature requires the integration of a Laplace equation. The problem of determining those systems of lines on the sphere for which the complete systems of surfaces corresponding can be obtained has been discussed by Darboux.[†] Treating the problem from a purely analytical standpoint, he reduces it to the determination of those Laplace equations having equal invariants which permit of particular solutions of modulus one, and which moreover can be integrated.[‡]

Considering the equations of this class, for which the sequence formed according to the method of Laplace terminates, and employing the analytic theory developed by Moutard,[§] Darboux shows that, from the solution of any one equation, can be obtained the solutions of all the others by simple quadratures.[||] From a purely analytical point of view, this treatment of the problem is most direct. The results, however, are not in such a form as to be easily incorporated with the other results of differential geometry, since the tangential coordinates alone are found, and since none of the functions appearing in the three fundamental forms for a surface are employed. Moreover, if one wishes to consider the isothermic surfaces which are solutions of the problem, it is impossible to do so from his treatment.[¶]

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[*] Read before the Amer. Math. Society, Chicago Section, April 18, 1908.

[†] Théorie des Surfaces, Vol. IV, p. 169.

[‡] *Loc. cit.*, p. 178.

[§] *Loc. cit.*, p. 181.

[||] *Loc. cit.*, p. 184.

[¶] Darboux realizes this fact, for he is obliged to use an entirely different treatment in determining the isothermic surfaces having one set of their lines of curvature plane. (See Vol. IV, Chapter X.)

We have considered the same problem in this paper from more of a geometric standpoint, taking as our point of departure the Gauss and Codazzi equations written in a form first used by Bonnet.* The functions involved are those which appear in the first and third form for a surface, together with two others which are closely related to them. Particular cases can be treated with great facility, and the characteristic forms for the linear elements may be written out immediately.† The systems of isothermic surfaces which are solutions of the problem, following this method, are easily treated by themselves.

We begin in §2 by explaining the meaning of the expression "*characteristic equation*" as used in this paper. In §3, we consider analytically the general class of surfaces which are solutions of the problem of the spherical representation. We have made use of the general theory as developed by Darboux‡ in order to show the relation between the systems of surfaces corresponding to particular solutions of the general characterizing equation.

In §4, we have called attention to a large class of surfaces belonging to the problem of the spherical representation which have their linear elements in a particular form.

The isothermic surfaces have been treated in §5, where it is shown that the general class may be obtained by simple algebraic processes combined with the integration of equations which are linear in a function of u and a function of v. In particular, we have pointed out that all isothermic surfaces having one set of their lines of curvature spherical can be obtained in this way from isothermic surfaces having one set of their lines of curvature plane.

In §6 and §7, we have considered the surfaces characterized respectively by two equations which play an important rôle in the general discussion. Among these, are the general class having one set of their lines of curvature plane or spherical. An important fact which we have noted in the first of these two articles — namely, that all isothermic surfaces, having one set of their lines of curvature plane, must have the same spherical representation of their lines of curvature as the Bonnet surfaces — makes unnecessary the discussion of these surfaces, to which Darboux has devoted the tenth chapter of volume four of the Théorie des Surfaces.

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* See reference to Bonnet's work given below.

† See §6 and §7.

‡ See previous references.

§ 2. *Characteristic Equations of Certain Classes of Surfaces.*

The Gauss and Codazzi equations for a surface referred to lines of curvature may be written as follows:

$$\left.\begin{aligned}&\frac{\partial M}{\partial v}-\frac{\partial N}{\partial u}-PQ=0,\\&\frac{\partial P}{\partial v}+MQ=0,\\&\frac{\partial Q}{\partial u}-NP=0,\end{aligned}\right\}\tag{1*}$$

where

$$M=-\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v},\quad N=\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u},\tag{2}$$

and where the first and third forms for the surfaces are respectively

$$\overline{ds}^2=E\overline{du}^2+G\overline{dv}^2,\tag{I}$$

$$\overline{d\sigma}^2=P^2\overline{du}^2+Q^2\overline{dv}^2.\tag{III}$$

In the system (1) are three equations involving four functions. If a fourth equation involving one or more of these functions is added, a certain system of lines on the Gauss sphere, in general, will be determined. We shall say that the equation added *defines the lines on the sphere*, and moreover, since only certain surfaces may have these as the spherical representation of their lines of curvature, we shall say that it is *characteristic of these surfaces.*

1°. For example, let us take with the system (1) the equation

$$M=0.\tag{3}$$

The linear element on the Gauss sphere takes the form

$$\overline{d\sigma}^2=\overline{du}^2+\phi(u,v)\,\overline{dv}^2,$$

where the form of the function ϕ is easily determined. Hence the lines $v=$ const. are geodesics or great circles. We say that equation (3) *defines on the sphere all systems composed of great circles and their orthogonal trajectories.*

Moreover, the first form for the corresponding surfaces may be written

$$\overline{ds}^2=du^2+\psi(u,v)\,dv^2,$$

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* Bonnet's form for these equations. Darboux sometimes uses them in this form, but not in the problem of the spherical representation.

where ψ has a certain form. Hence the lines $v=$ const. on these are geodesics. We say therefore that (3) is *characteristic of those surfaces which have one set of their lines of curvature geodesics.* Geodesic lines of curvature are of course plane lines.

2°. If the equation

$$M+N=0 \tag{4}$$

is used with the system (1), a very general system of lines is defined on the Gauss sphere. The first and third forms for the surfaces characterized by this equation may be written thus:*

$$\overline{ds}^2=\frac{\partial\phi}{\partial u}\,du^2+\frac{\partial\phi}{\partial v}\,dv^2, \tag{I}$$

$$\overline{d\sigma}^2=\frac{\partial\psi}{\partial u}\,du^2+\frac{\partial\psi}{\partial v}\,dv^2, \tag{III}$$

where ϕ and ψ are functions of u and v which must be chosen so as to satisfy the Gauss and Codazzi equations.

Belonging to this very general class of surfaces, we shall in a later article, § 4, call attention to a large system which is a solution of the problem of the spherical representation.

3°. The equation

$$\frac{\partial M}{\partial u}+\frac{\partial N}{\partial v}=0, \tag{5}$$

used with (1), defines upon the Gauss sphere *those systems of lines which permit of isothermic surfaces having them as the spherical representation of their lines of curvature.* For the linear element of any isothermic surface, if the parameters be properly chosen, may be written thus:

$$\overline{ds}^2=\lambda\,(du^2+dv^2),$$

and from (2) it appears then that (5) is satisfied. The equation (5) plays an important rôle in the discussion of the isothermic surfaces which are solutions of the problem of the spherical representation.†

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* For $\dfrac{\partial E}{\partial v}=\dfrac{\partial G}{\partial u}$.

† See § 5.

§ 3. *The Problem of the Spherical Representation.*

Of the defining equations noted in the preceding article, only the first, $M=0$, enables one to obtain the *general solution* of the systems of equations (1) and (2). There are, however, infinitely many equations which permit of this, as we may see as follows.

Combining the last two equations of (1) and solving the second for Q, we have, in the place of this system,

$$\left.\begin{aligned}&\frac{\partial M}{\partial v}-\frac{\partial N}{\partial u}-PQ=0,\\ &\frac{\partial^2 P}{\partial u\,\partial v}-\frac{\partial\log M}{\partial u}\frac{\partial P}{\partial v}+MNP=0,\\ &Q=-\frac{\frac{\partial P}{\partial v}}{M}.\end{aligned}\right\}\qquad(6)$$

The system (2), likewise, may be replaced by

$$\left.\begin{aligned}&\frac{\partial^2\sqrt{E}}{\partial u\,\partial v}-\frac{\partial\log M}{\partial u}\frac{\partial\sqrt{E}}{\partial v}+MN\sqrt{E}=0,\\ &\sqrt{G}=-\frac{\frac{\partial\sqrt{E}}{\partial v}}{M}.\end{aligned}\right\}\qquad(7)$$

The second equation in (6) is a Laplace equation in P. If M and N have such values that this equation can be solved, the corresponding solution of the system, as we shall see, can be obtained. Such a solution determines the lines on the Gauss sphere, and the linear element of the general class of surfaces having these as the spherical representation of their lines of curvature, can be obtained at once from (7).*

The whole problem then hinges upon the solution of a Laplace equation, the general solution of which, excluding a few particular cases, can be obtained when, and only when, one or both of the sequences, formed for it according to the method of Laplace, terminates; that is, leads to an equation for which one of the invariants vanishes.†

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* The Laplace equations in (6) and (7) are of course the same, but the solution of (6) is conditioned by the Gauss equation, whereas that of (7) is unconditioned.

† Darboux, *loc. cit.*, Vol. II, p. 28.

Calling the invariants of the Laplace equation in (6) h and k,[*] we have:

$$\begin{aligned} h &= -MN, \\ k &= -\frac{\partial^2 \log M}{\partial u\,\partial v} - MN, \\ k_1 &= -\frac{\partial^2 \log (Mk)}{\partial u\,\partial v} - k, \\ k_2 &= -\frac{\partial^2 \log (Mk\,k_1)}{\partial u\,\partial v} - k_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ k_{i+1} &= -\frac{\partial^2 \log (Mk \ldots . k_i)}{\partial u\, dv} - k_i, \end{aligned}$$

where k_i is the principal invariant for the corresponding equation of the sequence formed for the original equation in the one way.

The equation obtained by equating to zero any one of these invariants, taken with (6), *defines a system of lines on the Gauss sphere, and characterizes the general class of surfaces having these as the spherical representation of their lines of curvature.*[†]

Suppose we take as the defining equation

$$k_{i+1} = 0, \tag{8}$$

where i is arbitrarily large. Assuming that M and N have such values that (8) is satisfied, we can obtain P and Q from the last two equations of (6) at once in terms of M and N and arbitrary functions of integration.

Substituting in the first of (6), we have an equation in M and N which, used with (8), will determine them.

The values of E and G, which may now be found directly from (7), make known the linear element of the general class of surfaces. It is quite clear, however, that the actual determination of the surfaces in this way for any large value of i would be a very difficult, if not impossible problem. We shall point out, therefore, *a method for determining the general class of surfaces defined by* (8) *from those characterized by* $M = 0$.[‡]

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[*] Darboux, *loc. cit.*, Vol. II, p. 80.

[†] The problem of the spherical representation as we have considered it in this paper has to do with such surfaces only.

[‡] Similar to Darboux's results as geometrically interpreted by him (see *loc. cit.*, Vol. IV, p. 181).

It is clear that the lines on the sphere defined by $M = 0$[*] are included as a particular case in the class defined by $k = 0$, and that those defined by the latter are included as a particular case in the class defined by $k_1 = 0$. In general, then, *the lines defined by* $k_i = 0$ *are a part of the class defined by* $k_{i+1} = 0$, *and hence the surfaces characterized by the former are included in the class characterized by the latter equation.*

It follows from Darboux's work[†] that *the lines defined by* $k_{i+1} = 0$, *which are not included in the class defined by* $k_i = 0$, *are the spherical representation of the lines of curvature of the inverse with regard to a sphere of certain surfaces defined by the latter equation.* Hence the surfaces characterized by $k_{i+1} = 0$, which are not included among those characterized by $k_i = 0$, are *the inverse of certain of these*, or *surfaces which have the same spherical representation of their lines of curvature as these surfaces obtained by inversion.* To obtain, therefore, the general class of surfaces characterized by (8) we proceed as follows.

Taking the inverse of the general class defined by $M = 0$, we obtain by algebraic methods[‡] a new set of values for P and Q, and hence of M and N by (2). This new set of values is the general solution of (6) and $k_1 = 0$.[§] The linear element of the most general class of surfaces characterized by $k = 0$ is obtained directly by the integration of (7). Since the direction cosines of the normal to the surfaces (that is, the Cartesian coordinates X, Y, Z of a point on the Gauss sphere) are known, and since the principal radii of curvature r_1, r_2 can be obtained by division, the Cartesian coordinates for the surfaces characterized by $k = 0$ are given by the integration of the following equations:

$$\frac{\partial x}{\partial u} = r_2 \frac{\partial X}{du}, \quad \frac{\partial x}{\partial v} = r_1 \frac{\partial X}{\partial v}, \text{ etc.}$$

Taking the inverse of the surfaces thus found and repeating the operation, we obtain the Cartesian coordinates of the general class characterized by $k_1 = 0$. Repetition, finally, leads to the surfaces characterized by (8) for any value of i.

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[*] The equation $h = 0$ may be replaced by $M = 0$ without geometrical limitation, evidently.

[†] See previous references, and in particular Vol. IV, p. 181.

[‡] We are assuming that the Cartesian coordinates of the surfaces characterized by $M = 0$ have been obtained. See also Darboux, *loc. cit.*, Vol. II, p. 247.

[§] Must be, according to Darboux's work. See references above.

§ 4. *Surfaces of the Problem of the Spherical Representation Characterized by $M+N=0$.*

In the preceding article we assumed that the Laplace equation in (6) was such that the sequence formed in the one way was finite. Had we assumed that both sequences were finite (that is, that M and N had such values that, for some value of i, $h_i = k_i = 0$), we would have found that the ratio of M to N must be a function of u over a function of v, or choosing parameters properly,

$$M+N=0.$$

Hence, writing $-M$ in place of N in (6), we have as the defining equation to be used with it

$$\frac{\partial^2 \log (M k k_1 \ldots . k_i)}{\partial u \, \partial v} + k_i = 0, \tag{8'}$$

where $k k_1 \ldots . k_i$ are defined as in § 3.

We gave in § 2 the general form which the linear element of the surfaces characterized by (8′) takes. The surfaces obtained by inversion from a given class of these do not in general belong to the general class defined by (8′), but to the more general class characterized by (8). We can easily determine the necessary form* of the expression for the linear element of isothermic surfaces included here. For from § 2 it appears that

$$\frac{\dfrac{\partial \phi}{\partial u}}{\dfrac{\partial \phi}{\partial v}} = \frac{f(u)}{\phi(v)},$$

and hence, by a change of parameters,

$$\overline{ds}^2 = F(u+v)\left(\frac{du^2}{U} + \frac{dv^2}{V}\right).$$

§ 5. *Isothermic Surfaces.*

We found in § 2 that if the lines on the Gauss sphere are to permit of isothermic surfaces, then for a proper choice of parameters M and N must satisfy the equation (5). *The isothermic surfaces belonging to the problem of the*

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[*] This form is not characteristic of these surfaces. The linear element for the quadrics comes under the same form. The quadrics, however, might be considered a solution of the problem, since the linear element for the general class of surfaces having the same spherical representation of their lines of curvature can be obtained, as the Laplace equation can be integrated, though not by the method of Laplace.

spherical representation can be determined easily by algebraic operations and simple integration from those characterized by $M=0$. To prove this, let us note first that *all isothermic surfaces belonging to the class characterized by $k_{i+1}=0$, which do not at the same time belong to the class of surfaces characterized by $k_i=0$, must have the same spherical representation of their lines of curvature as the inverse of some isothermic surface* belonging to $k_i=0$.* For all lines defined by $k_{i+1}=0$, which do not also belong to $k_i=0$, are the spherical representation of the lines of curvature of *some* surface belonging to the class characterized by the latter; and for any surface *other than an isothermic surface* this would require that a function θ already uniquely determined as follows:

$$\theta=[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]^2,$$

where x, y, z and α, β, γ are respectively the coordinates of a point on the surface which is to be inverted, and of the pole of inversion, should necessarily be a solution of the equation †

$$\frac{\partial}{\partial v}\left(\frac{\theta}{\sqrt{E}}\frac{\partial\left(\frac{\sqrt{G}}{\theta}\right)}{\partial u}\right)-\frac{\partial}{\partial u}\left(\frac{\theta}{\sqrt{G}}\frac{\partial\left(\frac{\sqrt{E}}{\theta}\right)}{\partial v}\right)=0, \qquad (9)$$

where E and G are the functions in the expression for the linear element of the original surfaces.‡

Hence, to obtain the isothermic surfaces included in the class characterized by $k_{i+1}=0$, we begin with the isothermic surfaces included in the general class of surfaces $M=0$. These are surfaces of revolution and certain developable surfaces.

Taking their inverse with regard to a sphere, we have the "Bonnet surfaces",§ which of course belong to the class characterized by $k=0$. If we let the linear element of all other isothermic surfaces belonging to this class be written in the form

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* The inverse of an isothermic surface is of course an isothermic surface, and hence the corresponding lines on the sphere are defined by (5) and permit of such.

† See equation (5) and latter part of § 2.

‡ Of course it is possible that such a relation does exist in a very particular case; but it is easily shown that it does not exist for any known surfaces, and it is quite probable that there is no exception to the rule. If $E=G$, as is the case for isothermic surfaces, then (9) vanishes identically, irrespective of the values of the functions.

§ A name given to these surfaces because they were first determined by Bonnet. (*Journal de l'École Polytechnique*, Vol. XLII, pp. 182–151.)

$$\overline{ds}^2 = \lambda \left(\frac{du^2}{U} + \frac{dv^2}{V} \right),$$

where λ is a function of u and v to be determined, it follows that

$$M = -\sqrt{\frac{V}{U}} \frac{\partial \log \sqrt{\lambda}}{\partial v}, \quad N = \sqrt{\frac{U}{V}} \frac{\partial \log \sqrt{\lambda}}{\partial u},$$

where M *and* N *are formed for the Bonnet surfaces and hence are known.* The condition for integrability gives

$$\frac{\partial}{\partial u}\left(M\sqrt{\frac{U}{V}}\right) + \frac{\partial}{\partial v}\left(N\sqrt{\frac{V}{U}}\right) = 0, \tag{10}$$

an equation which is linear in U and V, and hence can in general be integrated.* The values of these functions being thus determined, the function λ is obtained immediately by quadrature.

As the direction cosines of the normal for the Bonnet surfaces are known, and as the principal radii of curvature r_1 and r_2 are obtained by division from the functions found, it follows that the Cartesian coordinates for the isothermic surfaces characterized by $k = 0$ are given at once by quadrature.† Beginning with the class of surfaces just determined and proceeding exactly as before, we obtain the isothermic surfaces‡ characterized by $k_1 = 0$, and from these those characterized by $k_2 = 0$, and so on, till finally we come to those characterized by $k_{i+1} = 0$ whatever the value of i.

This general method for determining the isothermic surfaces connected with the problem of the spherical representation has not, I believe, been pointed out before. The only difficulty to be met with is in the integration of (10) for the different values of M and N. An immediate application of the theory§ would be in the determination of all isothermic surfaces which have one set of their lines of curvature spherical. These, as is easily shown (see § 6 and § 7), have the same spherical representation of their lines of curvature as the inverse of the isothermic surfaces having one set of their lines of curvature plane.

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* We have already determined these surfaces by this method in a paper "On a Certain Class of Isothermic Surfaces" (*Transactions of the Amer. Math. Society*, Vol. X, pp. 79–94).

† See latter part of § 3.

‡ These include as a particular case all isothermic surfaces having one set of their lines of curvature spherical.

§ This theory will apply whether the isothermic surfaces are connected with the problem of the spherical representation proper or not. For suppose we have given any class of isothermic surfaces. The integration of (10) will give *all* isothermic surfaces having the same spherical representation, and inversion gives a new class with new lines on the sphere. The integration of (10) again applied to the new class would in general lead to a still more general class, etc.

§ 6. *Surfaces Characterized by $k = 0$.*

As we have already pointed out, *all surfaces characterized by the equation*

$$\frac{\partial^2 \log M}{\partial u \, \partial v} + MN = 0 \tag{11}$$

have the same spherical representation of their lines of curvature as the surfaces characterized by $M = 0$, and the inverse of these. Integrating, we find, from (6) and (7),[*] that the first and third forms for these surfaces may be written thus:

$$\overline{ds}^2 = M^2\left[\int \frac{f(u)\,du}{M} + \phi(v)\right]^2 du^2 + \frac{1}{M^2}\left[\frac{\partial}{\partial v}\left(M\int \frac{f(u)\,du}{M} + \phi(v)\right)\right]^2 dv^2, \tag{I}$$

$$\overline{d\sigma}^2 = M^2\left[\int \frac{U\,du}{M} + V\right]^2 du^2 + \frac{1}{M^2}\left[\frac{\partial}{\partial v}\left(M\int \frac{U\,du}{M} + V\right)\right]^2 dv^2, \tag{III}$$

where in (I) M, and hence the lines on the sphere, are entirely independent of the arbitrary functions[†] of integration $f(u)$ and $\phi(v)$, but in (III) M is a solution of an equation (the Gauss equation) in which the functions U and V appear.

The lines $v = \text{const.}$ on the surfaces $M = 0$ (see § 2) are plane lines, and the system of planes in which the lines lie cuts the surfaces at right angles.[‡] The lines $v = \text{const.}$ on the inverse of these surfaces, therefore, will be, in general, spherical lines, and, moreover, lines which have constant geodesic curvature, since the systems of planes will invert into systems of spheres cutting the surfaces at the original angle; that is, at right angles.

The necessary and sufficient condition that the lines on the surfaces corresponding to (I) *shall have constant geodesic curvature along the lines $v = \text{const.}$ is, that the function $f(u)$ shall vanish.*[§] The value of this function, however, does not affect the lines on the sphere, as we have already noted;[‖] and hence, as we would expect from the general theory, *all the surfaces characterized by* (11) *have the same spherical representation of their lines of curvature as those having constant geodesic curvature along the lines $v = \text{const.}$*

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[*] Assuming that M and N satisfy (11).

[†] These are the functions appearing in the general integral of the Laplace equation in (7).

[‡] A line of curvature having constant geodesic curvature, not zero, is the intersection of the surface and a sphere cutting at right angles. If the curvature is zero, the line is a geodesic line and the sphere is a plane.

[§] For $\frac{M}{\sqrt{E}}$ must equal a function of v. (See Bianchi, Vol. I, p. 210.)

[‖] See second note to § 3. This is merely an arbitrary function of the integral of the Laplace equation in (7).

It does not follow by any means that all the surfaces having constant geodesic curvature along the lines $v = \text{const.}$ are obtained by inversion from those characterized by $M = 0$, but merely that they all have the same spherical representation of their lines of curvature as these.

Now, it is easily shown* that all surfaces having one set of the lines of curvature plane are included in the class (11); and since the lines on the sphere which are the spherical representation of the lines of curvature must also be plane, it appears that in (III), for this particular class, U must vanish,† and hence the form (I), as written, and (III), with $U = 0$, are characteristic of these surfaces.

We can now state the theorem, *that all isothermic surfaces having one set of their lines of curvature plane must have the same spherical representation of their lines of curvature as the Bonnet surfaces.* We have already proved this theorem, as we showed above that all isothermic surfaces characterized by (11) have the same spherical representation of their lines of curvature as the Bonnet surfaces,‡ and as we have now just pointed out that all surfaces having one set of the lines of curvature plane are included in the system (11). This theorem reduces to one of a very elementary nature, the *problem of determining all isothermic surfaces having one set of the lines of curvature plane,* a problem which was solved in its most general form by Darboux,§ a particular case being solved later by P. Adam.‖ As we shall show¶ (§7) that all isothermic surfaces having one set of their lines of curvature spherical can be obtained from the isothermic surfaces having one set plane, the determination of these latter surfaces has been made a somewhat easier problem.

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* Expressing the fact that the lines on the Gauss sphere must have constant geodesic curvature, we find that M and N must satisfy (11).

† For $\frac{M}{Q}$ must equal a function of v.

‡ We might prove this directly as follows. Differentiating the equation formed by expressing the fact that the expression (I) corresponds to isothermic surfaces and holding count of the fact at the same time that M, and hence the lines on the sphere, must be independent of the function $f(u)$, we find that, choosing parameters properly, $M = -\sqrt{\frac{V_1}{U_1}}\frac{1}{(u+v)}$; and $N = \sqrt{\frac{U_1}{V_1}}\frac{1}{(u+v)}$, using (11). But the only lines on the sphere corresponding to these values of M and N are those which are the spherical representation of the lines of curvature of the Bonnet surfaces. (See Bonnet, *loc. cit.*, p. 132.)

§ *Loc. cit.*, Vol. IV, Chapter X.

‖ *Annales scientifiques de l'École Normale supérieure*, 3^{e} série, t. X, p. 319, 1893.

¶ This follows almost directly from the relation shown between surfaces having plane and spherical lines and from the work of §5.

Before closing this article, let us note that the only surfaces having one set of their lines of curvature spherical, belonging to (11), are those which have constant geodesic curvature along them.

§ 7. *Surfaces Characterized by* $k_1 = 0$.

The surfaces characterized by (11) may be divided into three classes as follows:

1°. Those having one set of their lines of curvature plane (corresponding to (11), when $U = 0$).

2°. Those having constant geodesic curvature along one set of the lines of curvature (corresponding to (11), when $f(u) = 0$ and $U \neq 0$).

3°. Surfaces having the same spherical representation of their lines of curvature as those in 2° (corresponding to (11), when $f(u) \neq 0$ and $U \neq 0$).

We shall now consider the surfaces obtained by inversion from these three classes, and the other surfaces which have the same spherical representation of their lines of curvature as these. As a whole, they would be characterized by the equation $k_1 = 0$, or

$$\frac{\partial^2 \log(Mk)}{\partial u\, \partial v} + k = 0, \tag{12}$$

where $k = -\dfrac{\partial^2 \log M}{\partial u\, \partial v} - MN$. Their first and third forms by (6) and (7), holding account of (12), may be written thus:

$$\begin{aligned}\overline{ds}^2 = M^2\left[\int k\left(\int \frac{f(u)}{Mk}\,du + \phi_1(v)\right)du + \phi_2(v)\right]^2 du^2 \\ + \frac{1}{M^2}\left[\frac{\partial}{\partial v}\left\{M\left[\int k\left(\int \frac{f(u)}{Mk}\,du + \phi_1(v)\right)du + \phi_2(v)\right]\right\}\right]^2 dv^2,\end{aligned} \tag{I}$$

$$\begin{aligned}\overline{d\sigma}^2 = M^2\left[\int k\left(\int \frac{U\,du}{Mk} + V_1\right)du + V_2\right]^2 du^2 \\ + \frac{1}{M^2}\left[\frac{\partial}{\partial v}\left\{M\left[\int k\left(\int \frac{U\,du}{Mk} + V_1\right)du + V_2\right]\right\}\right]^2 dv^2,\end{aligned} \tag{III}$$

where, as before, M and also k are independent of the functions appearing in (I), but dependent upon those appearing in (III), since the latter appear in an equation which M and k must satisfy. The surfaces obtained by inversion from the class 1° will, in general, have the lines $v =$ const. spherical. These belong to the class characterized by (12),[*] when $k \neq 0$. The surfaces which are the inverse of 2° all belong to the class characterized by (11), for either they must have the lines $v =$ const. plane and hence belong to the class (11) or else they have constant geodesic curvature and belong to the same.

It appears, therefore, *that the only surfaces characterized by* (12), *when* $k \neq 0$, *which have one set of their lines of curvature spherical are the inverse of* 1°, *together with certain surfaces having the same spherical representation of their lines of curvature.*[†] We can easily show to what form (I) and (III) reduce for surfaces having one set of their lines of curvature spherical. For Bianchi[‡] has shown that the necessary and sufficient condition that the lines $v =$ const. shall be spherical is that the principal radius of curvature r_2 shall be defined as follows:

$$r_2 = \alpha + \frac{\beta}{PQ} \frac{\partial P}{\partial v},$$

where α and β are functions of v. Applying this condition to the expression for r_2[§] obtained from (I) and (III), we can easily show by differentiation that both the functions $f(u)$ and U must vanish. Hence (III), with $U = 0$, corresponds to lines on the sphere which permit of surfaces having the lines $v =$ const. spherical. Moreover, *this is the linear element on the sphere for all surfaces having one set of their lines of curvature spherical, since the equation* $k_2 = 0$ *characterizes no surfaces having spherical lines* (*unless* $k_1 = 0$).

It follows, therefore, *that all surfaces having one set of their lines of curvature spherical have the same spherical representation of their lines of curvature as the inverse of the general class having one set of their lines of curvature plane.* This is a theorem which has been proved before in an entirely different way.[‖]

Collecting results, we see that the surfaces characterized by (12) and corresponding to (I) and (III) may be classified as follows:

1°. Surfaces having the lines $v =$ const. spherical (corresponding to the case where $f(u) = U = 0$).

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[*] Having spherical lines, they would not belong to (11), for they could not have constant geodesic curvature along them, since the spheres into which the planes invert would cut the surfaces at the original angle, which could not be 90° unless $M = 0$.

[†] Darboux has discussed surfaces having one set of their lines of curvature spherical (*loc. cit.*, Vol. IV, Chapter XI), following his general treatment of the problem of the spherical representation. Our results, so far as they can be compared, are in full agreement with his.

[‡] *Loc. cit.*, p. 304.

[§] $r_2 = \frac{\sqrt{E}}{P}$.

[‖] Darboux, *loc. cit.*, Vol. IV, p. 252.

2°. Surfaces having the same spherical representation of their lines of curvature as those of the preceding class (corresponding to the case $U = 0$ and $f(u) \neq 0$).

3°. Surfaces which are the inverse of the third class characterized by (11), and the others which have the same spherical representation of their lines of curvature as these (corresponding to the general case where $f(u) \neq 0$, $U \neq 0$).

The inverse of class 1° would give no new surfaces, but the inverse of 2°, and of 3° in part, would lead to other surfaces characterized by $k_2 = 0$. The general class of surfaces characterized by this equation is composed of these, of course, and the others having the same spherical representation of their lines of curvature.

In closing this paper, we will call attention to the ease with which the question as to whether any known surfaces are solutions of the problem of the spherical representation can be settled. If either the first or third forms are known, the functions M and N can be formed immediately, and substitution in the equation $k_{i+1} = 0$ settles the question. From Darboux's treatment of the problem such a question could be answered only after the greatest difficulty.

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The General Circulation of the Atmosphere.

BY FRANCIS R. SHARPE.

§ 1. *Introduction.*

The difference in temperature between the equator and the poles of the earth tends to produce a movement of the atmosphere from the poles towards the equator at the earth's surface, and thus set up a steady circulation. The problem has been investigated by Ferrel* and Oberbeck.† The former did not attempt to make a complete analytical solution. The latter assumed that the atmosphere was incompressible but that it satisfied Boyle's law for a perfect gas, and he surrounded the atmosphere with a spherical boundary of undetermined height. In the present paper the known hydrodynamical equations for a viscous gas are first determined, using polar coordinates, by considering the flow of matter and momentum through a small volume. By finding the flow of energy another equation is obtained, which is used instead of the ordinary adiabatic assumption. This equation is equivalent to the equation of energy in the kinetic theory of gases but is here obtained independently of that theory. Making use of the fact that the height of the atmosphere is small compared with the earth's radius, an approximation to the motion is found for a non-rotating earth. The effect of the earth's rotation on the east and west motion is next considered, and then the resulting modification of the pressure gradient. Finally the solution which is obtained is compared with the observed facts.

§ 2. *Coordinates and Variables.*

In discussing the general circulation of the atmosphere on a spherical earth, it is convenient to use polar coordinates; namely: r, the distance of any point of the atmosphere from the earth's center; θ, the colatitude; and ϕ, the longitude. The variables which we wish to determine as functions of r, θ, ϕ, and t, the time, are: p, ρ, τ, the pressure, density, and absolute temperature; and u, v, w, the components of the velocity.

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* Ferrel: Recent Advances in Meteorology.

† Oberbeck: Papers on the Mechanics of the Earth's Atmosphere, translated by Cleveland Abbé.

The atmosphere is supposed to be a perfect gas, so that

$$p = R\rho\tau, \tag{1}$$

where R is a constant. Five other equations are necessary to determine the six variables p, ρ, τ, u, v, w. They are derived in the next three sections by considering the flow of matter, of momentum, and of energy.

§ 3. *The Flow of Matter.*

Consider the flow of matter through the six faces of a small volume $dr \,.\, r d\theta \,.\, r \sin\theta\, d\phi$ bounded by the spheres r, $r + dr$, the cones θ, $\theta + d\theta$, and the planes ϕ, $\phi + d\phi$. The rate of increase of the matter in this volume is

$$\frac{\partial \rho}{\partial t}(dr \,.\, r d\theta \,.\, r \sin\theta\, d\phi) = -\frac{\partial}{\partial r}(\rho u \,.\, r d\theta \,.\, r \sin\theta\, d\phi)\, dr$$
$$-\frac{\partial}{\partial \theta}(\rho v \,.\, r \sin\theta\, d\phi \,.\, dr)\, d\theta - \frac{\partial}{\partial \phi}(\rho w \,.\, dr \,.\, r d\theta)\, d\phi.$$

Dividing by $dr \,.\, r d\theta \,.\, r \sin\theta\, d\phi$, we have the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{r^2 \partial r}(r^2 \rho u) + \frac{\partial}{r \sin\theta\, \partial\theta}(\rho v \sin\theta) + \frac{\partial}{r \sin\theta\, \partial\phi}(\rho w) = 0. \tag{2}$$

§ 4. *The Flow of Momentum.*

Consider the flow of momentum through the same small volume. Momentum being a vector quantity it is necessary to take account of the changes in the direction of the components of the velocity as we consider the different faces of the polar element of volume. Thus u turns through the angle $d\theta$ on the plane ϕ and through $\sin\theta\, d\phi$ on the cone θ, while v and $v + dv$ meet at the angle $\cos\theta\, d\phi$ on the earth's axis.

The normal stresses on the faces P, Q, R, the tangential stresses S, T, U, and the external force F also generate momentum; moreover the stresses change their directions as we consider different faces. The rate of increase of momentum in the direction of r is therefore

$$\frac{\partial}{\partial t}(\rho u)\, dr \,.\, r d\theta \,.\, r \sin\theta\, d\phi =$$
$$-\frac{\partial}{\partial r}\{(\rho u \,.\, u + P)\, r \sin\theta\, d\phi \,.\, r d\theta\}\, dr - \frac{\partial}{\partial \theta}\{(\rho v \,.\, u + U)\, r \sin\theta\, d\phi \,.\, dr\}\, d\theta$$
$$-\frac{\partial}{\partial \phi}\{(\rho w \,.\, u + T)\, r\, d\theta\, dr\}\, d\phi + (\rho v \,.\, v + Q)\, r \sin\theta\, d\phi\, dr \,.\, d\theta$$
$$+ (\rho w \,.\, w + R)\, r\, d\theta\, dr \sin\theta\, d\phi + \rho F_r\, dr \,.\, r d\theta \,.\, r \sin\theta\, d\phi.$$

Hence we have the equation of motion:

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{r^2 \partial r}\{(\rho u\,.\,u + P)\,r^2\} + \frac{\partial}{r \sin\theta\,\partial\theta}\{(\rho v\,.\,u + U)\sin\theta\}$$

$$+ \frac{\partial}{r \sin\theta\,\partial\phi}(\rho w\,.\,u + T) - \frac{\rho v\,.\,v + Q + \rho w\,.\,w + R}{r} - \rho F_r = 0. \qquad (3)$$

Similarly,

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{r^2 \partial r}\{(\rho u\,.\,v + U)\,r^2\} + \frac{\partial}{r \sin\theta\,\partial\theta}\{(\rho v\,.\,v + Q)\sin\theta\}$$

$$+ \frac{\partial}{r \sin\theta\,\partial\phi}(\rho w\,.\,v + S) - \frac{\rho v\,.\,u + U - (\rho w\,.\,w + R)\cot\theta}{r} - \rho F_\theta = 0 \qquad (4)$$

and

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{r^2 \partial r}\{(\rho u\,.\,w + T)\,r^2\} + \frac{\partial}{r \sin\theta\,\partial\theta}\{(\rho v\,.\,w + S)\sin\theta\}$$

$$+ \frac{\partial}{r \sin\theta\,\partial\phi}(\rho w\,.\,w + R) + \frac{\rho w\,.\,u + T + (\rho w\,.\,v + S)\cot\theta}{r} - \rho F_\phi = 0. \qquad (5)$$

If $\frac{\partial\rho}{\partial t}$ is eliminated by means of (2), these equations can easily be identified with the ordinary form of the equations of motion in polar coordinates.

In Stokes' theory of viscosity the stresses P, Q, R, S, T, U are linear functions of the strains e, f, g, a, b, c; namely:

$$\left.\begin{aligned} P &= p + \tfrac{2}{3}\mu\delta - 2\mu e, \\ Q &= p + \tfrac{2}{3}\mu\delta - 2\mu f, \\ R &= p + \tfrac{2}{3}\mu\delta - 2\mu g, \\ S &= -\mu a, \\ T &= -\mu b, \\ U &= -\mu c; \end{aligned}\right\} \qquad (6)$$

where the dilatation $\delta = e + f + g$; the three extensions are:

$$\left.\begin{aligned} e &= \frac{\partial u}{\partial r}, \\ f &= \frac{\partial v}{r \partial\theta} + \frac{u}{r}, \\ g &= \frac{\partial w}{r \sin\theta\,\partial\phi} + \frac{u}{r}; \end{aligned}\right\} \qquad (7)$$

and the three shears:

$$\left.\begin{aligned} a &= \frac{\partial v}{r\sin\theta\,\partial\phi} + \frac{\partial w}{r\,\partial\theta} - \frac{w}{r}\cot\theta, \\ b &= \frac{\partial u}{r\sin\theta\,\partial\phi} + \frac{\partial w}{\partial r} - \frac{w}{r}, \\ c &= \frac{\partial u}{r\,\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}. \end{aligned}\right\} \qquad (8)$$

On substituting the values of the stresses in the equations of motion we could obtain them in the form which is generally used, but it is more convenient for our purpose to leave them in their present form.

§ 5. *The Flow of Energy.*

By considering the flow of matter and momentum we have obtained the equations of continuity and motion. In an exactly similar way the kinetic energy per unit of volume is $\frac{1}{2}\rho(u^2+v^2+w^2)$, and like ρ it is a scalar quantity. The energy in the form of heat is $C\rho\tau$, where C is the capacity of the atmosphere for heat. The rate of conduction of heat is $-\varkappa \operatorname{grad}\tau$, where $\varkappa$ is the conductivity. The stresses P, Q, R, S, T, U, and the external force F also do work and thus contribute energy. Taking account of all these different forms of energy, we find for the rate of increase of energy in a small polar element of volume:

$$\begin{aligned} &\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\rho(u^2+v^2+w^2)+C\rho\tau\right\}dr\,.\,r\,d\theta\,.\,r\sin\theta\,d\phi = \\ &\quad -\frac{\partial}{\partial r}\left[\left\{\tfrac{1}{2}\rho(u^2+v^2+w^2)u + C\rho\tau u + Pu + Uv + Tw - \varkappa\frac{\partial\tau}{\partial r}\right\}r\,d\theta\,r\sin\theta\,d\phi\right]dr \\ &\quad -\frac{\partial}{\partial\theta}\left[\left\{\tfrac{1}{2}\rho(u^2+v^2+w^2)v + C\rho\tau v + Uu + Qv + Sw - \varkappa\frac{\partial\tau}{r\,\partial\theta}\right\}r\sin\theta\,d\phi\,dr\right]d\theta \\ &\quad -\frac{\partial}{\partial\phi}\left[\left\{\tfrac{1}{2}\rho(u^2+v^2+w^2)w + C\rho\tau w + Tu + Sv + Rw - \varkappa\frac{\partial\tau}{r\sin\theta\,\partial\phi}\right\}r\,d\theta\,dr\right]d\phi \\ &\quad +\rho(F_r u + F_\theta v + F_\phi w)\,dr\,r\,d\theta\,r\sin\theta\,d\phi. \end{aligned}$$

Using (2) to eliminate $\frac{\partial\rho}{\partial t}$, substituting for the three components of F from (3), (4) and (5), and using (7) and (8), we find

$$C\rho\frac{d\tau}{dt}+Pe+Qf+Rg+Sa+Tb+Uc$$
$$-\frac{\partial}{r^2\partial r}\left(\varkappa r^2\frac{\partial\tau}{\partial r}\right)-\frac{\partial}{r\sin\theta\,\partial\theta}\left(\varkappa\frac{\partial\tau}{r\partial\theta}\sin\theta\right)-\frac{\partial}{r\sin\theta\,\partial\phi}\left(\varkappa\frac{\partial\tau}{r\sin\theta\,d\phi}\right)=0.$$

On substituting the values of the stresses from (6) this equation becomes

$$C\rho\frac{d\tau}{dt}+p\delta+\frac{2}{3}\mu\delta^2-2\mu\left(e^2+f^2+g^2\right)-\mu\left(a^2+b^2+c^2\right)$$
$$-\frac{\partial}{r^2\partial r}\left(\varkappa r^2\frac{\partial\tau}{\partial r}\right)-\frac{\partial}{r\sin\theta\,\partial\theta}\left(\varkappa\frac{\partial\tau}{r\partial\theta}\sin\theta\right)-\frac{\partial}{r\sin\theta\,\partial\phi}\left(\varkappa\frac{\partial\tau}{r\sin\theta\,\partial\phi}\right)=0.$$

This result agrees with the equation of energy in the kinetic theory of gases, but is here derived independently.

Replacing p and δ by the use of (1), (2) and (7), we obtain the more convenient form

$$C\rho\frac{d\tau}{dt}-R\tau\frac{d\rho}{d\tau}+\frac{2}{3}\mu\delta^2-2\mu\left(e^2+f^2+g^2\right)-\mu\left(a^2+b^2+c^2\right)$$
$$-\frac{\partial}{r^2\partial r}\left(\varkappa r^2\frac{\partial\tau}{\partial r}\right)-\frac{\partial}{r\sin\theta\,\partial\theta}\left(\varkappa\frac{\partial\tau}{r\partial\theta}\sin\theta\right)-\frac{\partial}{r\sin\theta\,\partial\phi}\left(\varkappa\frac{\partial\tau}{r\sin\theta\,\partial\phi}\right)=0. \quad (9)$$

The conductivity and viscosity of the atmosphere are both small; hence, if the motion is so slow that we may neglect squares of velocities, (9) becomes

$$C\rho\frac{d\tau}{dt}-R\tau\frac{d\rho}{d\tau}=0.$$

Integrating, we find that

$$\tau^{C}\propto\rho^{R},$$

which gives, from (1),

$$p\propto\rho^{\frac{R+C}{C}},$$

the adiabatic law.

Since R is about $.4\,C$, we have

$$p=m\rho^{1.4}, \quad (10)$$

where m is a constant.

§ 6. *Adiabatic Equilibrium.*

When the temperature is a function of r only, the equations of motion become

$$\frac{\partial p}{\partial r}+g\rho=0,$$

or, from (10),

$$1.4\, m\rho^{.4}\frac{d\rho}{dr}+g\rho=0.$$

Integrating, we have

$$\frac{1.4}{.4}\, m\rho^{.4}+gr=\text{constant};$$

that is,

$$\frac{1.4}{.4}\, R\tau+gr=\text{constant}.$$

At the surface of the earth r is a and τ is τ_0; therefore

$$\frac{1.4}{.4}\, R(\tau-\tau_0)+g(r-a)=0.$$

Hence τ is 0 when

$$r=a+\frac{1.4}{.4g}R\tau_0. \tag{11}$$

Substituting numerical values, we find that the height of the atmosphere is $\frac{1}{220}a$, which will be denoted by ah, where $h=\frac{1}{220}$.

If we now put $r=a(1+h-x)$, so that ax is the distance from the top of the atmosphere, then τ is a linear function of x which vanishes when $x=0$ and has the value τ_0 when $x=h$.

Hence, we have

$$\tau=\tau_0\left(\frac{x}{h}\right); \tag{12}$$

and therefore, from (1) and (10),

$$p=p_0\left(\frac{x}{h}\right)^{\frac{7}{2}} \tag{13}$$

and

$$\rho=\rho_0\left(\frac{x}{h}\right)^{\frac{5}{2}}. \tag{14}$$

A relation between the constants is furnished by (1) and (11); namely,

$$g=\frac{7p_0}{2ah\rho_0}. \tag{15}$$

§ 7. *Circulation on a Non-Rotating Earth.*

Assume that the temperature of the atmosphere decreases symmetrically from the equator to the poles and diminishes uniformly from the surface upwards. Then, instead of (11), we have

$$\tau = \frac{x}{h}(\tau_0 - \tau_2 \cos^2\theta), \tag{16}$$

where $\frac{\tau_2}{\tau_0}$ is small.

To satisfy (1), and from the form of the equations of motion, we therefore take instead of (13) and (14):

$$\left.\begin{aligned} p &= p_0\left(\frac{x}{h}\right)^{\frac{7}{2}} + p_2 \cos^2\theta, \\ \rho &= \rho_0\left(\frac{x}{h}\right)^{\frac{5}{2}} + \rho_2 \cos^2\theta, \end{aligned}\right\} \tag{17}$$

where

$$\frac{p_2}{p_0\left(\frac{x}{h}\right)^{\frac{7}{2}}} = \frac{\rho_2}{\rho_0\left(\frac{x}{h}\right)^{\frac{5}{2}}} - \frac{\tau_2}{\tau_0}. \tag{18}$$

Equilibrium is no longer possible, because, from (6) and (17),

$$\frac{\partial Q}{\partial \theta} \neq 0$$

and (4) is therefore not satisfied.

§ 8. *Solution of the Continuity Equation.*

When the motion is steady and symmetrical, the continuity equation takes the form

$$\frac{\partial}{r^2\partial r}(\rho r^2 u) + \frac{\partial}{r\sin\theta\,\partial\theta}(\rho v \sin\theta) = 0,$$

or approximately, from (14),

$$\left(\frac{x}{h}\right)^{\frac{5}{2}}\left\{-\frac{\partial u}{\partial x} + \frac{\partial}{\sin\theta\,\partial\theta}(v\sin\theta)\right\} - \frac{5}{2h}\left(\frac{x}{h}\right)^{\frac{3}{2}} u = 0.$$

Hence

$$\frac{\partial u}{\partial x} + \frac{5}{2}\,\frac{u}{x} = \frac{\partial}{\sin\theta\,\partial\theta}(v\sin\theta).$$

The solution of this equation, which corresponds to (17), is

$$\left.\begin{aligned} u &= U(1-3\cos^2\theta), \\ v &= V\sin\theta\cos\theta, \end{aligned}\right\} \tag{19}$$

where U and V are functions of x which satisfy

$$\frac{dU}{dx} + \frac{5}{2}\frac{U}{x} = -V. \tag{20}$$

§ 9. *Approximate Form of the Equations of Motion.*

Since x is a small ratio which lies between 0 and h (that is, between 0 and $\frac{1}{330}$), we see from (20) that U is small compared with V. Therefore, if we retain in the equations of motion only the most important terms, we find

$$\frac{\partial p}{\partial r} + g\rho = 0, \tag{21}$$

$$\frac{\partial p}{r\,\partial\theta} - \frac{\partial}{r^3\,\partial r}\left(\mu\frac{\partial v}{\partial r}r^3\right) = 0. \tag{22}$$

From (15), (17) and (19) these equations become

$$\frac{\partial p_2}{\partial x} = \frac{7}{2}\frac{p_0\rho_2}{h\rho_0} \tag{23}$$

and

$$2ap_2 = -\frac{\partial}{\partial x}\left(\mu\frac{\partial V}{\partial x}\right).$$

Since the viscosity varies as the .76 power of the absolute temperature, the latter equation may be written in the form

$$2ap_2 = -\frac{\partial}{\partial x}\left(\mu_0\left(\frac{x}{h}\right)^{\frac{3}{4}}\frac{\partial V}{\partial x}\right). \tag{24}$$

§ 10. *Solution of the Equations of Motion.*

Finding first p from (18) and (21), next V from (24), and then U from (20), we have

$$U = Ax^{-\frac{5}{2}} + Bx^{\frac{11}{4}} + Cx^{\frac{23}{4}}\log x + Dx^{\frac{5}{4}} + Ex, \tag{25}$$

where A, B, D and E are arbitrary constants, but C must be so chosen that (18) is satisfied.

§ 11. *The Boundary Conditions.*

At the top of the atmosphere (where x is zero) it is assumed that u is zero and that the tangential stress $\mu \frac{\partial v}{\partial r}$ is zero; hence A and D are zero.

At the surface of the earth u and v are both zero; hence

$$U = B\left(\frac{19}{4} x^{\frac{23}{4}} \log \frac{x}{h} - x^{\frac{23}{4}} + h^{\frac{19}{4}} x\right). \tag{26}$$

Substituting this value of U in (18), we find

$$u = \frac{896\, a\, p_0 \tau_2}{107217\, h^{\frac{11}{4}} \mu_0 \tau_0}\left(\frac{19}{4} x^{\frac{23}{4}} \log \frac{x}{h} - x^{\frac{23}{4}} + h^{\frac{19}{4}} x\right)(1 - 3 \cos^2 \theta), \tag{27}$$

$$v = -\frac{896\, a\, p_0 \tau_2}{107217\, p^{\frac{11}{4}} \mu_0 \tau_0}\left(\frac{627}{16} x^{\frac{19}{4}} \log \frac{x}{h} - \frac{7}{2} x^{\frac{19}{4}} + \frac{7}{2} h^{\frac{19}{4}}\right) \sin\theta \cos\theta, \tag{28}$$

$$p = p_0\left(\frac{x}{h}\right)^{\frac{7}{4}}\left\{1 + \left(\frac{7}{2} \log \frac{x}{h} + \frac{119}{99}\right) \cos^2 \theta\right\}, \tag{29}$$

$$\rho = \rho_0\left(\frac{x}{h}\right)^{\frac{5}{4}}\left\{1 + \left(\frac{7}{2} \log \frac{x}{h} + \frac{218}{99}\right) \cos^2 \theta\right\}. \tag{30}$$

§ 12. *East and West Motion on a Rotating Earth.*

When the rotation of the earth is taken into account, it is convenient to denote by w the east and west velocity relative to the earth so that the actual velocity is $w + \varepsilon r \sin\theta$, ε being the earth's angular velocity about its axis. Substituting this value in (5) and retaining only the most important terms, we have

$$0 = -2 \varepsilon \rho u \sin\theta - 2 \varepsilon \rho v \cos\theta + \frac{\partial}{\partial r}\left(\mu \frac{\partial w}{\partial r}\right);$$

hence, using (19),

$$\frac{\partial}{\partial x}\left(x^{\frac{1}{4}} \frac{\partial w}{\partial x}\right) = \frac{2 a^2 \varepsilon \rho_0}{\mu_0 h^{\frac{7}{4}}}\left\{U \sin\theta (1 - 3 \cos^2 \theta) + V \sin\theta \cos^2 \theta\right\}.$$

The term containing U is small, compared with the term containing V, except near the equator, where $\cos\theta$ is small. Substituting from (27) and (28) for U and V, integrating twice and remembering that at the top of the atmosphere (where x is zero) the tangential stress $\mu \frac{\partial w}{\partial r}$ is zero and that at the surface of the earth (where $x = h$) w is zero, we find

$$\left.\begin{aligned} w = \frac{1792\, a^3 \varepsilon p_0 \rho_0 \tau_2}{107217\, h^{\frac{9}{2}} \mu_0^2 \tau_0} \Bigg[\Bigg\{ & \frac{2}{37} x^{\frac{19}{2}} \log \frac{x}{h} - \frac{596}{19 . 37^2} x^{\frac{19}{2}} + \frac{8}{171} h^{\frac{19}{4}} x^{\frac{19}{4}} \\ & - \frac{5588}{171 . 37^2} h^{\frac{19}{2}} \Bigg\} \sin\theta\, (1 - 3\cos^2\theta) \\ - \Bigg\{ \frac{19}{34} x^{\frac{17}{2}} \log \frac{x}{h} - \frac{53}{289} x^{\frac{17}{2}} & + \frac{4}{15} h^{\frac{19}{4}} x^{\frac{15}{4}} - \frac{361}{4335} h^{\frac{17}{2}} \Bigg\} \sin\theta \cos^2\theta \Bigg]. \end{aligned}\right\} \quad (32)$$

§ 13. *Pressure Gradient Due to the East and West Motion.*

When the rotation of the earth is taken into account, the equations of motion (21) and (22) become

$$0 = -\frac{\partial p}{\partial r} - g\rho + 2\varepsilon\rho w \sin\theta, \qquad (33)$$

$$0 = -\frac{\partial p}{r\,\partial\theta} + 2\varepsilon\rho w \cos\theta + \frac{\partial}{r^2\,\partial r}\left(\mu r^2 \frac{\partial v}{\partial r}\right). \qquad (34)$$

We must therefore replace (17) by

$$\left.\begin{aligned} \rho &= \rho_0 \left(\frac{x}{h}\right)^{\frac{5}{2}} + \rho_2 \cos^2\theta + \rho_4 \cos^4\theta, \\ p &= p_0 \left(\frac{x}{h}\right)^{\frac{7}{2}} + p_2 \cos^2\theta + p_4 \cos^4\theta, \end{aligned}\right\} \qquad (35)$$

in which, from the law of gases, as in (18),

$$\frac{p_4}{p_0\left(\frac{x}{h}\right)^{\frac{7}{2}}} = \frac{\rho_4}{\rho_0\left(\frac{x}{h}\right)^{\frac{5}{2}}}. \qquad (36)$$

Again, (19) must be altered to

$$u = U(1 - 3\cos^2\theta) + U_4(3\cos^2\theta - 5\cos^4\theta),$$
$$v = V \sin\theta\cos\theta + V_4 \sin\theta\cos^3\theta,$$

where (from the continuity equation)

$$\frac{\partial U_4}{\partial x} + \frac{5}{2}\frac{U_4}{x} = -V_4. \qquad (37)$$

From (34) we see that p_4 is of the same order as ρw in x, while V_4 is of order two higher in x; and therefore, from (37), U_4 is of order three higher in x. Hence (33) becomes

$$0 = -\frac{\partial p}{\partial r} - g\rho.$$

Using (36) and (15) we therefore find, on integration,

$$p_4 = Px^{\frac{9}{2}},$$

where P is an arbitrary constant which will be subsequently determined so that the boundary conditions are satisfied.

Substituting in (34), we have

$$0 = 4Px^{\frac{7}{2}} \sin\theta \cos^3\theta + 2a\epsilon\rho w \cos\theta + \frac{\mu_0}{ah^{\frac{3}{4}}} \frac{\partial}{\partial x}\left(x^{\frac{3}{4}} \frac{\partial v}{\partial x}\right) \sin\theta \cos^3\theta.$$

Substituting the value of w from (32) and retaining only the second term in that equation, we have

$$0 = 4Px^{\frac{7}{2}} - \frac{3584\, a^4 \epsilon^2 p_0 \rho_0^2 \tau_2}{107217\, h^7 \mu_0^2 \tau_0} \left\{\frac{19}{34} x^{11} \log\frac{x}{h} - \frac{53}{289} x^{11} + \frac{4}{15} h^{\frac{13}{4}} x^{\frac{25}{4}} - \frac{361}{4335} h^{\frac{17}{4}} x^{\frac{5}{2}}\right\}$$
$$+ \frac{\mu_0}{ah^{\frac{3}{4}}} \frac{\partial}{\partial x}\left(x^{\frac{3}{4}} \frac{\partial V_4}{\partial x}\right).$$

Integrating with the boundary condition that, when x is zero, the tangential stress $\mu \frac{\partial v}{\partial x}$ is zero, we find

$$0 = \frac{8}{9} Px^{\frac{9}{2}} - \frac{3584\, a^4 \epsilon^2 p_0 \rho_0^2 \tau_2}{107217\, h^7 \mu_0^2 \tau_0} \left\{\frac{19}{408} x^{12} \log\frac{x}{h} - \frac{1595}{288 . 289} x^{12} + \frac{16}{435} h^{\frac{13}{4}} x^{\frac{29}{4}}\right.$$
$$\left. - \frac{722}{30345} x^{\frac{7}{2}} h^{\frac{17}{4}}\right\} + \frac{\mu_0}{ah^{\frac{3}{4}}} x^{\frac{3}{4}} \frac{\partial V_4}{\partial x}.$$

Dividing by $x^{\frac{3}{4}}$ and integrating with the boundary condition that, when x is h, V_4 is zero, we have

$$0 = \frac{32}{171} P\left(x^{\frac{19}{4}} - h^{\frac{19}{4}}\right) - \frac{3584\, a^4 \epsilon^2 p_0 \rho_0^2 \tau_2}{107217\, h^7 \mu_0^2 \tau_0} \left\{\frac{19}{4998} x^{\frac{49}{4}} \log\frac{x}{h} - \frac{93559}{72 . 17^2 . 49^2} x^{\frac{49}{4}}\right.$$
$$\left. + \frac{32}{4525} h^{\frac{13}{4}} x^{\frac{15}{2}} - \frac{2888}{455175} h^{\frac{17}{4}} x^{\frac{15}{4}} - \frac{87,822,265}{52200 . 17^2 . 49^2} h^{\frac{49}{4}}\right\} + \frac{\mu_0}{ah^{\frac{3}{4}}} V_4.$$

Substituting the value of V_4 in (37), multiplying by $x^{\frac{3}{4}}$, integrating with the boundary condition that U_4 is zero when x is zero, and, finally, putting x equal to h, we have, as the condition that U_4 is zero, the equation to determine P:

$$0 = \frac{32}{171}\,P\left(\frac{4}{33} - \frac{2}{7}\right)h^{\frac{33}{4}} - \frac{3584\,a^4\,\varepsilon^2 p_0\,\rho_0^2\,\tau_2}{107{\cdot}217\,h^7\,\mu_0^2\,\tau_0}\left\{-\frac{19}{4998}\,\frac{4^2}{63^2} - \frac{93559}{72.17^2.49^2}\,\frac{4}{63}\right.$$
$$\left. + \frac{32}{6525}\,\frac{1}{11} - \frac{2888}{455175}\,\frac{4}{29} - \frac{87,822,265}{52200.17^3.49^2}\,\frac{2}{7}\right\}h^{\frac{63}{4}}.$$

Hence

$$P = \frac{140.7\,a^4\,\varepsilon^2 p_0\,\rho_0^2\,\tau_2\,h^{\frac{1}{2}}}{107217\,\mu_0^2\,\tau_0},$$

and therefore

$$p = p_0\left(\frac{x}{h}\right)^{\frac{7}{2}}\left\{1 + \frac{\tau_2}{\tau_0}\left(\frac{7}{2}\log\frac{x}{h} + \frac{119}{99}\right)\cos^2\theta + \frac{140.7\,a^4\,\varepsilon^3 p_0\,\rho_0^2\,\tau_2\,h^{\frac{1}{2}}}{107217\,\mu_0^2\,\tau_0}\cos^4\theta\right\}. \quad (38)$$

§ 14. *Discussion of the Results Obtained.*

The vertical velocity u_2 has a maximum value

$$\frac{383\,a\,p_0\,\tau_2\,h^3}{107217\,\mu_0\,\tau_0}$$

when x is about $.55\,h$.

The southerly velocity v_2 has a maximum value

$$\frac{1023\,a\,p_0\,\tau_2\,h^2}{107217\,\mu_0\,\tau_0}$$

when x is about $.886\,h$; and it changes to northerly when x is $.73\,h$.

It is seen from (32) that the velocity is easterly except near the equator, where, on account of the smallness of $\cos\theta$, the second term is important and the velocity is westerly.

The pressure p_2 has its maximum positive value,

$$\frac{119}{99}\,\frac{p_0\,\tau_2}{\tau_0},$$

when x is h. It vanishes when x is $.71\,h$ and has its greatest negative value when x is $.53\,h$.

§ 15. *The Observed Facts in the Atmosphere.*

It is known that* the temperature at the surface of the earth is

$$8.5 - 20.95\,\cos^2\theta,$$

or, in our notation,

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* Ferrel: Recent Advances in Meteorology, p. 452.

$$\begin{aligned}\tau &= 303 - 42\cos^2\theta,\\ &= \tau_0 - \tau_2\cos^2\theta;\end{aligned}$$

so that $\frac{\tau_2}{\tau_0}$ is not very small. It is also known that[*]

$$p = 758 + 31\cos^2\theta - 61\cos^4\theta.$$

Putting $x = h$ in (38), the value obtained was

$$p = p_0 + \frac{119\,p_0\tau_2}{99\,\tau_0}\cos^2\theta - \frac{140.7\,a^4\epsilon^2 p_0\rho_0^2\tau_2 h^{\frac{1}{3}}}{107217\,\mu_0^2\tau_0}\cos^4\theta;$$

hence our value for p_2 appears to be too large, although its value decreases very rapidly as we ascend vertically. As regards p_4, if we substitute numerical values we find too large a value. The explanation is probably that, on account of the large amount of eddying motion in the atmosphere, the ordinary experimental value for μ is smaller than its effective value in the atmosphere. Moreover the motion we are considering is unstable except for small values of $\frac{\tau_2}{\tau_0}$ and large values of μ. There appears then to be only a qualitative and not a quantitative confirmation of the solution obtained.

As regards the velocities, it is agreed[†] that the velocity at the equator is westerly, and in the middle latitude easterly, and increasing with the altitude; but there is no evidence of a northerly motion in the upper atmosphere as theory would lead us to expect.

CORNELL UNIVERSITY, *July*, 1908.

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[*] Overbeck: *loc. cit.*

[†] Hildebrandson: Brit. Assoc. Report, 1908.

Generalizations of the Tetrahedral and Octahedral Groups.

By G. A. Miller.

§ 1. *Introduction.*

A generalization of these groups is included in an article entitled "Generalization of the Groups of Genus Zero."* In the present paper many of the results of this article are greatly extended, and hence this paper may be regarded to be, in part, a generalization of a generalization of the tetrahedral and the octahedral groups. The fundamental importance of the groups of genus zero adds interest to these extensions, and the simple character of these extensions seems to justify the hope that they may find extensive applications. The present paper has close contact with a recent paper entitled "Finite Groups Which may be Defined by Two Operators Satisfying Two Conditions."†

In what follows, it is frequently necessary to speak of a group of order 24 which does not include any subgroup of order 12. As there is only one such group, it is completely determined by stating its order and the fact that it does not involve a subgroup of order 12. To secure greater brevity we shall hereafter refer to this group as the non-twelve G_{24}, or the non-twelve group of order 24. In accord with this notation, the tetrahedral group may be characterized as the non-six G_{12}, and the icosahedral group could be distinguished by the fact that it is the non-invariant-five group of order 60, since the icosahedral group is the only group of order 60 which involves more than one subgroup of order 5. While this method of notation frequently leads to brevity of statement without any material loss in clearness, it is not to be inferred that it solves all the difficulties connected with distinguishing the different groups of the same order.

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* *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 1.

† American Journal of Mathematics, Vol. XXXI (1909), p. 167.

§ 2. *Generalizations of the Tetrahedral Group.*

It is well known that the tetrahedral group may be defined by each of the following sets of three equations:

$$s_1^3 = 1,\ s_2^3 = 1,\ (s_1 s_2)^2 = 1;\quad s_1^2 = 1,\ s_2^3 = 1,\ (s_1 s_2)^3 = 1.$$

Instead of the first of these sets, we shall consider the more general but closely related set of two equations:

$$s_1^3 = s_2^3,\quad (s_1 s_2)^2 = (s_2 s_1)^2.$$

By equating the second to unity there results a set of equations which were considered in the generalization mentioned above, while defining relations of the tetrahedral group are obtained when each of these two equations is made equal to unity. Hence these equations may be regarded as a double generalization of defining relations of the tetrahedral group.

From the equation $(s_1 s_2)^2 = (s_2 s_1)^2$ it is easy to derive the following four:

$$(s_1 s_2^2)^n = (s_2 s_1)^{2n} s_2^{-1} s_1^{-n} s_2,\quad (s_2 s_1^2)^n = (s_1 s_2)^{2n} s_1^{-1} s_2^{-n} s_1,$$
$$(s_1^2 s_2)^n = (s_2 s_1)^{2n} s_1 s_2^{-n} s_1^{-1},\quad (s_2^2 s_1)^n = (s_1 s_2)^{2n} s_2 s_1^{-n} s_2^{-1}.$$

These equations are useful to simplify some of the relations given below. The group $\{s_1, s_2\} = G$, generated by s_1, s_2, involves s_1^3 and $(s_1 s_2)^2$ invariantly, since each of these operators is commutative with both s_1 and s_2. As $s_1 s_2$ and $s_2 s_1$ have a common square, the product of one of these operators into the inverse of the other is transformed into its inverse by each of them.* This property will be used to find an upper limit for the orders of some of the operators in G, as follows:

$$s_1 s_2 \cdot s_1^{-1} s_2^{-1} = s_1^{-6} s_1 \cdot s_2 s_1^2 \cdot s_2^2 = s_1^{-6} (s_1 s_2)^2 s_2^{-1} s_1 s_2^2 = s_1^{-6} (s_1 s_2)^2 s_2^{-1} \cdot s_1 s_2 \cdot s_2,$$
$$(s_1 s_2 s_1^{-1} s_2^{-1})^3 = s_1^{-12} (s_1 s_2)^4 s_2^{-1} (s_1 s_2)^2 s_2 = s_1^{-12} (s_1 s_2)^6.$$

As $s_1^{-12} (s_1 s_2)^6$ is both invariant under $s_1 s_2$ and is also transformed into its inverse by this operator, its order is either 2 or 1.

Having found an upper limit for the order of $s_1^{-12} (s_1 s_2)^6$, it is easy to prove that the order of

$$Q = \{s_1^{-6} (s_1 s_2)^2 s_2^{-1} \cdot s_1 s_2 \cdot s_2 = s_1^{-6} (s_2^{-1} s_1 s_2^2)^3,\ s_1^{-6} (s_2 s_1)^3,\ s_1^{-6} (s_1 s_2)^3\}$$

divides 8 and that Q is invariant under G. The three given generators of Q form a complete set of conjugates under s_2, and hence Q is invariant under s_2. These generators have a common square, since the square of one is invariant under G. The product of the last into the inverse of the second has been proved to be equal to the first. As it has also been proved that the order of the first divides 4, it results that the order of Q must divide 8. As the last two given generators of Q are transformed among themselves by s_1, Q is invariant under s_1 as well as under s_2; that is, Q is invariant under G. Moreover, Q involves the commutator subgroup of G, since it includes

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* *Archiv der Mathematik und Physik*, Vol. IX (1905), p. 6.

$$s_1^{-6}(s_2 s_1)^3 \cdot s_1^6 (s_1 s_2)^{-3} = s_2 s_1 s_2^{-1} s_1^{-1}.$$

Hence every subgroup of G which involves Q is invariant under G.

When s_1, s_2 are commutative, the second relation $(s_1 s_2)^2 = (s_2 s_1)^2$ does not give any additional information, and hence G is any abelian group which may be generated by two operators having a common cube. That is, G may be any cyclic group, or the direct product of such a group and the group of order 3, whenever s_1, s_2 are commutative. This disposes of the case when s_1, s_2 are commutative, and we shall therefore assume that these operators are non-commutative during the rest of the discussion relating to the equations

$$s_1^3 = s_2^3, \quad (s_1 s_2)^2 = (s_2 s_1)^2.$$

As s_1, s_2 are non-commutative, the order of Q cannot be less than 4, and hence Q is either the quaternion group or the four group as the order of $s_1^{-12}(s_1 s_2)^6$ is 2 or 1. Since $(s_1 s_2)^4 s_1^{-8}$ is of order 3, it results from the above that $\{Q, (s_1 s_2)^4 s_1^{-8}\}$ is the non-twelve group of order 24 or the tetrahedral group as the order of $s_1^{-12}(s_1 s_2)^6$ is 2 or 1. It is also evident that Q is actually the commutator subgroup of G. This proves the theorem: *If two non-commutative operators satisfy the two conditions $s_1^3 = s_2^3$, $(s_1 s_2)^2 = (s_2 s_1)^2$, they generate a group whose commutator subgroup is either the tetrahedral group or the non-twelve group of order* 24.

The abelian group $\{s_1^3, (s_1 s_2)^2\}$ and the commutator subgroup of G clearly generate G, and these two groups have only the identity in common when the commutator subgroup is the tetrahedral group, since this involves no invariant operator besides the identity. That is, when the commutator subgroup of G is the tetrahedral group, G is the direct product of $\{s_1^3, (s_1 s_2)^2\}$ and $\{Q, (s_1 s_2)^4 s_1^{-8}\}$. When the commutator subgroup of G is the non-twelve group of order 24, it has exactly two operators in common with $\{s_1^3, (s_1 s_2)^2\}$.

Although the conditions $s_1^3 = s_2^3$, $(s_1 s_2)^2 = (s_2 s_1)^2$ are not sufficient to limit the order of G, they restrict G to a very elementary category of solvable groups. All of these groups are isomorphic to the tetrahedral group with respect to an abelian invariant subgroup; viz., $\{s_1^3, (s_1 s_2)^2\}$. In other words, the quotient group of G with respect to this abelian subgroup is the tetrahedral group. By assigning a finite value either for the order of s_1^3 or for the order of $(s_1 s_2)^2$, there results a finite number of possible groups. The possible groups for one such case, viz., when the order of $(s_1 s_2)^3$ is 1, were considered in the article mentioned above with the heading, "Generalization of the Groups of Genus Zero." By employing the results given above, it would be an easy matter to determine all the possible groups in other cases.

The second generalization of the tetrahedral group may be represented by the two equations

$$s_1^2 = s_2^3, \quad (s_1 s_2)^3 = (s_2 s_1)^3.$$

Defining relations of the tetrahedral group may evidently be obtained by making each member of these equations equal to unity. To obtain an upper limit for the orders of some of the operators in G we may proceed as follows:

$$s_1, \quad s_2^{-1} s_1 s_2, \quad s_2^{-2} s_1 s_2^2 = s_2 s_1 s_2^{-1}$$

are three operators of G, and they have a common square since s_1^2 is invariant under G. From

$$s_2^{-1} s_1 s_2 \cdot s_2 s_1^{-1} s_2^{-1} = (s_2^{-1} s_1^{-1})^2 s_2^5 = (s_1 s_2)^{-3} s_1^5$$

it results that $(s_1 s_2)^{-6} s_1^{10}$ is both invariant under $s_2^{-1} s_1 s_2$ and is also transformed into its inverse by this operator. Hence $(s_1 s_2)^{-6} s_1^{10}$ is of order 2 or 1.

The group

$$\{(s_1 s_2)^{-3} s_1^5, \ (s_1 s_2)^{-3} s_2^{-1} s_1^5 s_2, \ (s_1 s_2)^{-3} s_2 s_1^5 s_2^{-1}\} = Q$$

is invariant under s_2 since it is generated by a complete set of conjugates under s_2, and it is also transformed into itself by s_1 because it is transformed into itself by each of the three operators $(s_1 s_2)^{-3} s_1^5$, $(s_1 s_2)^3$, s_1^4. The last two of these operators are evidently invariant under G. As $(s_1 s_2)^{-3} s_1^5$ is transformed into its inverse by $s_2^{-1} s_1 s_2$, it must also be transformed into its inverse by $(s_1 s_2)^{-3} s_2^{-1} s_1^5 s_2$. Moreover, the three given generators of Q have a common square, and it has been proved that the product of the second into the inverse of the third is equal to the first. Hence it results that Q is the quaternion group or the four group as the order of $(s_1 s_2)^{-6} s_1^{10}$ is 2 or 1. It is also easy to see that Q is the commutator subgroup of G, since it includes

$$(s_1 s_2)^3 s_1^{-5} \cdot (s_1 s_2)^{-3} s_2^{-1} s_1^5 s_2 = s_1^{-1} s_2^{-1} s_1 s_2 = (s_1 s_2)^{-3} s_2 s_1^5 s_2^{-1}.$$

Since G is generated by Q, s_2, $(s_1 s_2)^3$, it is isomorphic with the tetrahedral group with respect to the abelian subgroup $\{s_1^2, (s_1 s_2)^3\}$. From what precedes there results the following theorem: *If two non-commutative operators satisfy the conditions* $s_1^2 = s_2^3$, $(s_1 s_2)^3 = (s_2 s_1)^3$, *they generate a group G whose commutator subgroup is either the quaternion group or the four group and whose central is* $\{s_1^2, (s_1 s_2)^3\}$. *The quotient group of G with respect to its central is the tetrahedral group.*

In the case considered above when $s_1^3 = s_2^3$, $(s_1 s_2)^2 = (s_2 s_1)^2$, it was possible to prove that G involves a non-invariant operator of order 3 and hence such a G includes the tetrahedral group or the non-twelve group of order 24. In the present case it is not possible to prove the existence of such subgroups. In fact, it is not difficult to see that s_1, s_2 may be so selected that the order of every non-invariant operator whose order is a power of 3 exceeds any finite number. If s_1, s_2 are any two operators which are commutative with t and satisfy the relations $s_1^2 = s_2^3$, $(s_1 s_2)^3 = (s_2 s_1)^3$, the operators $t^a s_1$, $t s_2$ will also satisfy these relations, provided $t^3 = t^{2a}$. By selecting t in such a way that its order is an arbitrary odd number divisible by 3, the given relations may be satisfied, and hence it is possible to select t so as to make the order of every operator of G which is not in the group generated by its central and its commutator subgroup exceed any finite number.

Suppose that the order of $s_1 s_2$ is $3m$, m being prime to 3. Since $(s_1 s_2)^{12} = s_1^{10}$, the order of s_1 must be prime to 3. As s_1 must be of even order when G is non-abelian, we may assume that the order of s_1 is $2k$ and hence the order of s_2 is $3k$. Hence s_2^k is a non-invariant operator of order 3 and $\{s_2^k, Q\}$ is either the tetrahedral group or the non-twelve group of order 24. This group is invariant under G, as it involves the commutator subgroup of G. It is evident that G is generated by the two groups $\{s_1^4, (s_1 s_2)^3\}$ and $\{s_2^k, Q\}$, and that these groups have at most two common operators. If $(s_1 s_2)^3$ is of odd order, the former of these groups must be of odd order, and hence G is the direct product of these groups. Hence the theorem: *If two non-commutative operators satisfy the relations* $s_1^2 = s_2^3$, $(s_1 s_2)^3 = (s_2 s_1)^3$ *and if* $(s_1 s_2)^3$ *is of an odd order prime to* 3, *then these operators generate a group which is the direct product of the cyclic group generated by* s_1^4 *and either the tetrahedral group or the non-twelve group of order* 24 *as the order of* $(s_1 s_2)^{-6} s_1^{10}$ *is* 1 *or* 2. This includes the known theorem as regards the groups generated by two operators satisfying the relations $s_1^2 = s_2^3$, $(s_1 s_2)^3 = 1$.*

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* *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 4.

In what precedes, it was assumed that s_1, s_2 are not commutative. If they are commutative, the relation $(s_1 s_2)^3 = (s_2 s_1)^3$ does not imply any further restriction and hence G is cyclic, being generated by two commutative operators which satisfy the relation $s_1^\alpha = s_2^\beta$, where α and β are relatively prime. That is, if two commutative operators satisfy the relations $s_1^2 = s_2^3$, $(s_1 s_2)^3 = (s_2 s_1)^3$, they generate a cyclic group and every cyclic group may be generated by two such operators.

§ 3. *Generalizations of the Octahedral Group.*

By assuming that each member of the equations

$$s_1^2 = s_2^4, \quad (s_1 s_2)^3 = (s_2 s_1)^3$$

is equal to the identity, we arrive at well-known defining relations of the octahedral group. Hence these relations may be regarded as a double generalization of the octahedral group, and it is our first object to study the fundamental properties of the system of groups which may be generated by two non-commutative operators satisfying these general relations. As in the cases considered in the preceding section, we shall first find an upper limit for the orders of some operators contained in such a group G.

The three operators

$$s_2^2, \quad s_2^{-1} s_1^{-1} s_2^2 s_1 s_2, \quad s_1 s_2^2 s_1^{-1}$$

constitute a complete set of conjugates under $s_1 s_2$. As they have a common square, any two of them generate the generalized dihedral group. The product of the third into the inverse of the second is

$$\begin{aligned} s_1 s_2^2 s_1^{-1} s_2^{-1} s_1^{-1} s_2^{-2} s_1 s_2 &= s_1 s_2^2 (s_2 s_1)^{-2} s_2^{-1} s_1 s_2 = (s_1 s_2)^{-3} s_1 s_2^3 s_1 s_2^{-1} s_1 s_2 \\ &= (s_1 s_2)^{-3} (s_1 s_2^3)^3 s_2^{-6} = (s_1 s_2)^{-6} s_2^{18}. \end{aligned}$$

As $(s_1 s_2)^{-12} s_2^{36}$ is both invariant under G and also transformed into its inverse by $s_1 s_2^2 s_1^{-1}$, its order is 2 or 1, and $(s_1 s_2)^{24} = s_2^{72}$.

The group

$$\{(s_1 s_2)^{-6} s_2^{18}, \ s_2^{-1} s_1^{-1} s_2^{18} s_1 s_2 (s_1 s_2)^{-6}, \ s_1 s_2^{18} s_1^{-1} (s_1 s_2)^{-6}\} = Q$$

is invariant under $s_1 s_2$, since it is generated by a complete set of conjugates under this operator. It is also invariant under s_1, since s_1 transforms the first and last of its generators among themselves and these two operators generate Q, as the first has been proved equal to the product of the third into the inverse of the second. From the last fact it results also that Q is the quaternion group or the four group as the order of $(s_1 s_2)^{-12} s_2^{36}$ is 2 or 1. As Q is invariant under both $s_1 s_2$ and s_1, it is invariant under G.

From the fact that the non-invariant operator

$$s_2^{24}(s_2 s_1)^{-8}$$

is of order 3, it results that $\{Q, s_2^{24}(s_2 s_1)^{-8}\}$ is either the tetrahedral group or the non-twelve group of order 24. We proceed to prove that this is the commutator subgroup of G. In fact, this results from the following product of operators contained in $\{Q, s_2^{24}(s_2 s_1)^{-8}\}$:

$$s_2^{18}(s_2 s_1)^{-6} . s_2^{24}(s_2 s_1)^{8} = s_2^{-6}(s_2 s_1)^2 = s_2^{-1} s_1^{-1} s_2 s_1 .$$

As $\{Q, s_2^{24}(s_2 s_1)^{-8}\}$ is the commutator subgroup of G, it results that every other subgroup of G which involves this subgroup is invariant under G. In particular, the commutator subgroup and $s_2^{9}(s_1 s_2)^{-3}$ generate an invariant subgroup of G whose order is twice the order of this commutator subgroup. This invariant subgroup is the symmetric group of order 24 if $(s_1 s_2)^{-12} s_2^{36}$ is the identity. If this operator is of order 2, it is a known group of order 48 involving the non-twelve group of order 24. The central of G is evidently $\{s_1^2, (s_1 s_2)^8\}$ and G is generated by its central and the group $\{Q, s_2^{24}(s_2 s_1)^{-8}, s_2^{9}(s_1 s_2)^{-3}\}$. Hence the theorem: *If two non-commutative operators satisfy the conditions* $s_1^2 = s_2^4$, $(s_1 s_2)^8 = (s_2 s_1)^8$, *they generate a group whose commutator subgroup is either the tetrahedral group or the non-twelve group of order* 24, *and whose central is generated by* s_1^2, $(s_1 s_2)^8$. *The quotient group with respect to this central is the symmetric group of order* 24, *and the entire group is generated by its central and a group whose order divides* 48.

As G is evidently generated by the abelian group $\{s_1^8, (s_1 s_2)^8\}$ and the group $\{Q, s_2^{24}(s_2 s_1)^{-8}, s_2^{9}(s_1 s_2)^{-3}\}$ whose order divides 48, it results that G is the direct product of these groups whenever $s_1 s_2$ is of odd order. In this case $\{s_1^8, (s_1 s_2)^8\}$ is the cyclic group generated by s_1^8, and hence it is easy to determine all the possible groups in which $(s_1 s_2)^8$ has for its order a given odd number. The particular case when this order is unity has been considered, and the results constitute a special case of what has just been proved.*

A second generalization of the octahedral group is given by the equations

$$s_1^2 = s_2^3, \quad (s_1 s_2)^4 = (s_2 s_1)^4.$$

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* *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 6.

In this case we shall consider the transforms of $(s_1 s_2)^2$ with respect to s_2. The complete set of these transforms is

$$(s_2 s_1)^2, \quad (s_1 s_2)^2, \quad (s_2^{-1} s_1 s_2^2)^2.$$

The product of the second into the inverse of the third is

$$\begin{aligned} s_1 s_2 s_1 s_2^{-1} s_1^{-1} s_2^{-1} s_1^{-1} s_2 &= s_1 s_2 s_1^2 (s_2 s_1)^{-8} s_2^2 = (s_1 s_2)^{-4} s_1 s_2 s_1^2 . s_2 s_1 . s_2^2 \\ &= (s_1 s_2)^{-4} s_1 s_2^2 s_1 s_2^2 s_1^2 = (s_2 s_1)^{-6} s_1^{10}. \end{aligned}$$

Since $(s_2 s_1)^{-12} s_1^{20}$ is both invariant under G and also transformed into its inverse by $(s_1 s_2)^2$, its order is 2 or 1.

It will now be proved that the group

$$\{(s_2 s_1)^{-6} s_1^{10},\ (s_1 s_2)^{-6} s_1^{10},\ (s_2^{-1} s_1 s_2^2)^{-6} s_1^{10}\} = Q$$

is invariant under G and that it is either the four group or the quaternion group. From the equations given above, it results directly that the product of the second generator of Q and the inverse of the third is the first, and hence Q is generated by any two of the given set. Since the first of these generators is transformed into its inverse by the second and since they have a common square, it results that Q is the quaternion group or the four group as $(s_1 s_2)^{-12} s_1^{20}$ is of order 2 or 1. As Q is generated by a complete set of conjugates under s_2, it is invariant under this operator. It is also invariant under s_1, since s_1 transforms the first two generators among themselves. This completes the proofs of the facts that Q is invariant under G and that it is either the quaternion group or the four group.

The operator $(s_2 s_1)^{-8} s_2^{20}$ is a non-invariant operator of order 3, and hence $\{Q, (s_2 s_1)^{-8} s_2^{20}\}$ is the tetrahedral group or the non-twelve group of order 24 as $(s_2 s_1)^{-12} s_2^{30}$ is of order 1 or 2. This is also the commutator subgroup (C) of G, since it includes

$$(s_1 s_2)^8 s_2^{-20} . (s_1 s_2)^6 s_2^{15} = (s_1 s_2)^2 s_2^{-5} = s_1 s_2 s_1^{-1} s_2^{-1}$$

and C can clearly not be a subgroup of $\{Q, (s_2 s_1)^{-8} s_2^{20}\}$. Having proved that $C = \{Q, (s_2 s_1)^{-8} s_2^{20}\}$, it results directly that this group is invariant under G. The quotient group of G with respect to its central $\{s_1^2, (s_1 s_2)^4\}$ is the symmetric group of order 24. In the preceding generalization of the octahedral group it was possible to prove that G involves a group of order 24 or of order 48, which has an $(\alpha, 1)$ isomorphism with this quotient group. In the present generalization, on the contrary, it is not possible to prove the existence of any finite group in G having a $(\beta, 1)$ isomorphism with the quotient group with respect to the central, since there is no limit to the highest power of 2 that may occur in the order of every operator of G not contained in the group generated by its central and its commutator subgroup.

It is evident that $\{s_1^6, (s_1 s_2)^8, (s_2 s_1)^{-8} s_2^{20}\} \equiv \{s_1^2, s_2\}$, and that this is the group generated by the central and the commutator subgroup of G. If it is assumed that s_1^2 is of odd order, it results that $s_1^{2k} = s_1^{40}$, k being an odd number, and hence $s_1^k (s_1 s_2)^{-12}$ is an operator of order 2 which is not contained in $\{s_1^2, s_2\}$. By adjoining $s_1^k (s_1 s_2)^{-12}$ to $\{Q, (s_2 s_1)^{-8} s_2^{20}\}$ there results a group which has an $(\alpha, 1)$ isomorphism with the octahedral group, where $\alpha = 2$ or 1. In this case the abelian group $\{s_1^6, (s_1 s_2)^8\}$ is clearly of odd order, and G is the direct product of this abelian group and the group $\{Q, (s_2 s_1)^{-8} s_2^{20}, s_1^k (s_1 s_2)^{-12}\}$. Hence the theorem: *If two non-commutative operators satisfy the equations* $s_1^2 = s_2^3$, $(s_1 s_2)^4 = (s_2 s_1)^4$, *they generate a group whose commutator subgroup is either the tetrahedral group or the non-twelve group of order* 24. *If* s_1^2 *is of odd order, the group* $\{s_1, s_2\}$ *is the direct product of an abelian group of odd order and a group whose order divides* 48.

The third generalization of the octahedral group is given by the equations

$$s_1^3 = s_2^4, \quad (s_1 s_2)^2 = (s_2 s_1)^2.$$

In the preceding section we deduced four useful formulas directly from the latter of these equations. These formulas will be used in what follows. Just as in the preceding two cases, we shall prove that the commutator subgroup of G is either the tetrahedral group or the non-twelve group of order 24. The steps towards this proof may be taken as follows:

$$s_2^{-2} \,.\, s_1^{-2} s_2^2 s_1^2 = s_2^{-8} (s_2^2 s_1)^2 s_1 = s_2^{-8} (s_1 s_2)^4 s_2 s_1^{-2} s_2^{-1} s_1 = s_2^{-12} (s_1 s_2)^6 s_1^{-1} s_2^{-2} s_1 .$$

Hence $(s_2^{-2} s_1^{-2} s_2^2 s_1^2)^2 = s_2^{-28} (s_1 s_2)^{12}$. As this operator is both invariant under G and also transformed into its inverse by s_2^2, its order is 2 or 1.

The group

$$Q = \{s_2^{-12} (s_1 s_2)^6 s_1^{-1} s_2^{-2} s_1,\ s_2^{-12} (s_1 s_2)^6 s_1 s_2^{-2} s_1^{-1},\ s_2^{-12} (s_1 s_2)^6 s_2^{-2}\}$$

is generated by a complete set of conjugates under s_1, and the product of the third generator and the inverse of the second has been proved equal to the first, since $s_1 s_2^{-2} s_1^{-1} = s_1^{-2} s_2^{-2} s_1^2$. It is also evident that the second and the third of these generators transform the first into its inverse and that these three generators have a common square. Hence Q is the quaternion group or the four group as $s_2^{-28} (s_1 s_2)^{12}$ is of order 2 or 1. Since Q is invariant under s_1 and also under $s_2 s_1$, it must be invariant under G.

As $s_1^{-14}(s_1s_2)^8$ is a non-invariant operator of order 3, the group $\{Q, s_1^{-14}(s_1s_2)^8\}$ is either the tetrahedral group or the non-twelve group of order 24. It is also the commutator subgroup of G, since it includes

$$s_1^{-14}(s_1s_2)^8 \cdot s_2^{14}(s_1s_2)^{-6} = s_1^{-2}(s_1s_2)^2 s_2^{-2} = s_1^{-1}s_2s_1s_2^{-1}$$

and cannot be larger than the commutator subgroup. The central of G $\{s_1^3, (s_1s_2)^2\}$ and this commutator subgroup clearly generate $\{s_1, s_2^2\}$. As in the preceding case it is impossible to assign an upper limit for the minimum order of an operator of G which is not contained in this subgroup, since the condition $s_1^3 = s_2^4$ implies that such an order may be divisible by an arbitrary power of 2.

If we assumed that $(s_1s_2)^2$ is of odd order, it results that $(s_1s_2)^{2k} = (s_1s_2)^{24}$, k being an odd number. Hence $s_2^{-28}(s_1s_2)^k$ is an operator of order 2 which is not found in $\{s_1, s_2^2\}$. The group $\{Q, s_1^{-14}(s_1s_2)^8, s_2^{-28}(s_1s_2)^k\}$ is invariant, and its order is twice that of the commutator subgroup of G. This group and the abelian group of odd order $\{s_1^6, (s_1s_2)^2\}$ must generate G; and as the latter is composed of operators of odd order, G is the direct product of these two groups. Hence the theorem: *If two non-commutative operators satisfy the equations* $s_1^3 = s_2^4$, $(s_1s_2)^2 = (s_2s_1)^2$, *and if the order of* $(s_1s_2)^2$ *is odd, the group generated by these operators is the direct product of an abelian group of odd order* $\{s_1^6, (s_1s_2)^2\}$ *and a group whose order divides* 48. *The latter has an* $(\alpha, 1)$ *isomorphism with the octahedral group, where* $\alpha = 1$ *or* 2. *Only a finite number of groups is possible whenever the order of one of the operators* s_1, s_2, s_1s_2 *is given.*

In what precedes, it was assumed that s_1, s_2 are non-commutative. If these operators are commutative, each of these three cases becomes very simple since the second equations do not imply any further restriction. Hence it results that two commutative operators which satisfy separately the three pairs of equations

$$s_1^2 = s_2^4,\ (s_1s_2)^3 = (s_2s_1)^3; \quad s_1^2 = s_2^3,\ (s_1s_2)^4 = (s_2s_1)^4; \quad s_1^3 = s_2^4,\ (s_1s_2)^2 = (s_2s_1)^2$$

generate in the first case either any cyclic group or the direct product of such a group and the group of order 2, while in each of the other two cases they can generate only cyclic groups.

The Theory of Degenerate Algebraical Curves and Surfaces.

By O. E. Glenn.

Chasles[*] arrived by geometrical intuition at the fundamentals of the theory of characteristics of plane curves.

An important principle at the basis of this theory is the fact that a system of curves defined by definite conditions will in general contain degenerate curves which, though degenerate, still satisfy all of the defining conditions of the system. It was this fact that led Cayley[†] to emphasize the importance of the general question of the degeneracy of curves as an object of investigation. This he did in an article "Sur les courbes aplaites" in volume 74 of the Comptes Rendus, and subsequently in a few notes to be found in his Works. The subject has nevertheless never been exhaustively investigated. A few important contributions have been published, and the question has been studied more or less incidentally in a number of places, but it would seem that much yet remains to be done.

As an application of some of his researches on symmetric functions, Junker[‡] studied in 1894 the decomposition of forms into linear factors. Hočevar[§] has treated the linear and quadratic factors of forms, and his work has been extended recently by O. Dorner.[‖] Other references to work in this field are the following:

H. M. Taylor: "On the Decomposition of a Cubic Curve," *Proceedings London Mathematical Society*, Vol. XXVIII.

Meyer: "Zur Theorie der reduciblen ganzen Functionen von n Variablen," *Math. Annalen*, Bd. XXX.

Gordan: "Das Zerfallen der Curven in gerade Linien," *Math. Annalen*, Bd. XLV.

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[*] Chasles: *Comptes Rendus*, t. 64, pp. 799, 1079.

[†] Cayley: *Comptes Rendus*, t. 74, p. 708.

[‡] Junker: *Math. Annalen*, Bd. XLV, S. 1–84.

[§] Hočevar: *Wiener Sitzungsberichte*, 1904 and 1907.

[‖] Dorner: "Ueber Teiler von Formen," *Progr. Königsberg*, 1908.

Brill: "Ueber die Zerfallung der Ternärformen in linearen Factoren," *Math. Annalen*, Bd. L.

Hadamard: "Sur les conditions de decomposition d'une forme ternaire," Bordeaux, *Proces-Verbaux*, 1897.

Most of the published articles treat the formal side of the question only. I have* in this paper sought to develop methods which would throw light on the geometrical phases of the theory, and to interpret various algebraical conditions which cause degeneracy. The analytical theory forming the first part of the paper is essentially a generalization of certain results of Junker in his memoir, previously quoted, on symmetric functions of several groups of variables.

Cayley introduced the name "penultimate curve" for the locus whose equation contains certain infinitesimal coefficients the vanishing of which renders the equation factorable into an ultimate form

$$P^{\alpha} Q^{\beta} R^{\gamma} \ldots . = 0.$$

We shall also use the term "penultimate surface" for the locus of a degenerating quaternary form.

PART I.

THE ASYMPTOTIC FORMS OF A GIVEN FORM.

§ 1. *A Class of Symmetric Functions.*

Let the m roots of

$$f(x) = x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots . + a_m = 0$$

be $x_1, x_2, \ldots., x_m$. Let the totality of the sums of the roots taken in pairs, and of the products of the roots in pairs be respectively

$$\xi_i, \ \eta_j \ [i, j = 1, 2, \ldots., \tfrac{1}{2} m (m-1)].$$

The elementary symmetric functions of the ξ_i and of the η_j can be evaluated in terms of the coefficients a of $f(x)$. For

$$\Sigma \xi_1 \xi_2 \ldots . \xi_k = \phi_k = J_k \ (x_1, x_2, \ldots., x_m), \tag{1}$$

$$\Sigma \eta_1 \eta_2 \ldots . \eta_k = \psi_k = I_k \ (x_1, x_2, \ldots., x_m) \quad (k = 1, 2, \ldots., m), \tag{2}$$

where I and J are symmetric in the x's.

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[*] The reviewer who may wish a summary of the principal results of the paper may read those parts which have been printed in italics.

For illustration, if $f(x)$ has rational coefficients and is resolvable into quadratic factors in the absolute field, it can be resolved by factoring $\phi_{(m)}$ and a_m, trying the factors, as ξ, η, in $x^2 - \xi x - \eta$ and retaining those quadratics which are found to divide $f(x)$. Any equation can be tested for this resolvability by a finite number of trials.*

Generalization.

Let the totality of sums of the roots of $f(x)$ taken ν in a sum, of the sums of products two at a time of the roots of each of these sums, of the sums of products three at a time of the roots of each of these original sums, and so on, be respectively represented by

$$\xi_{1i},\ \xi_{2j},\ \xi_{3k},\ \ldots\ \ [i, j, k, \ldots = 1, 2, \ldots, C_\nu^m].$$

Then we have

$$\begin{aligned}
&\Sigma\, \xi_{11}\xi_{12} \ldots \xi_{1h} = J_{1h}(x_1, x_2, \ldots, x_m) = \phi_{1h}(a),\\
&\Sigma\, \xi_{21}\xi_{22} \ldots \xi_{2h} = J_{2h}(x_1, x_2, \ldots, x_m) = \phi_{2h}(a),\\
&\Sigma\, \xi_{31}\xi_{32} \ldots \xi_{3h} = J_{3h}(x_1, x_2, \ldots, x_m) = \phi_{3h}(a) \quad [h = 1, 2, \ldots, C_\nu^m],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}$$

where J_{gh} is symmetric in the x's, and hence $\phi_{gh}(a)$ rational in the a's. *In order that*

$$a_x^\nu = x^\nu - \sum_{g=1}^{\nu} \xi_g x^{\nu - g}$$

may be a divisor of $f(x)$, ξ_g *must be a root of*

$$\Phi_{g\xi}^{C_\nu^m} = \xi^{C_\nu^m} + \phi_{g1}\,\xi^{C_\nu^m - 1} + \phi_{g2}\,\xi^{C_\nu^m - 2} + \ldots + \phi_{gC_\nu^m} \tag{5}$$

and conversely. For example, when $m = 5$, $\nu = 4$, $h = 1$,

$$\begin{aligned}
\phi_{11} &= 4a_1,\\
\phi_{12} &= 6a_1^2 + a_2,\\
\phi_{13} &= 4a_1^3 + 3a_1a_2 - a_3,\\
\phi_{14} &= a_1^4 + 3a_1^2a_2 - 2a_1a_3 + a_4,\\
\phi_{15} &= a_1^3a_2 - a_1^2a_3 + a_1a_4 - a_5.
\end{aligned}$$

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* Let $f(x) = x^4 + 2x^2 + x + 2 = 0$. Then ξ divides $a_1a_2a_3 - a_1^2a_4 - a_3^2 = -1$; $\xi = \pm 1$, while η divides $a_4 = 2$; $\eta = \pm 1, \pm 2$. By trial we retain the pairs $(\xi, \eta) = (-1, -1), (1, -2)$; *i. e.*, $f(x) = (x^2 + x + 1)(x^2 - x + 2)$.

§ 3. *Quadratic Factors of Forms.*

The decomposition of forms into linear factors has been studied by Junker, who gave a method of determining the component forms, and discovered relations which exist among the coefficients of the form when it is degenerate. Let us consider forms which are decomposable into quadratic factors.

Let the ternary form

$$f_{8m}(x, y, 1) = a_{0m}y^m + a_{1m-1}y^{m-1}x + \dots + a_{m0}x^m + a_{0m-1}y^{m-1} \\ + \dots + a_{01}y + a_{10}x + a_{00}$$

equal the product of the $\frac{1}{2}m$ quadratic forms

$$\sigma_i = s_i x^2 + t_i xy + u_i y^2 + v_i x + w_i y + r_i \qquad (i = 1, 2, \dots, \tfrac{1}{2}m).$$

Then we have

$$\begin{aligned}
&\Sigma\, s_1 s_2 \dots s_g = a_{m0}, \\
&\Sigma\, s_1 s_2 \dots s_{g-1} t_g = a_{m-11}, \\
&\Sigma\, s_1 s_2 \dots s_{g-1} u_g + \Sigma\, s_1 s_2 \dots s_{g-2} t_{g-1} t_g + \dots = a_{m-22}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\
&\Sigma\, u_1 u_2 \dots u_g = a_{0m}, \\
&\Sigma\, s_1 s_2 \dots s_{g-1} v_g = a_{m-10}, \\
&\dots\dots\dots\dots\dots, \\
&\Sigma\, u_1 u_2 \dots u_{g-1} w_g = a_{0m-1} \qquad [g = \tfrac{1}{2}m],
\end{aligned}$$

from which we get the following differential relations:

$$\sum \frac{\partial a_{m-kk}}{\partial s_1} v_1 + \sum \frac{\partial a_{m-kk}}{\partial t_1} w_1 = a_{m-k-1k} \qquad (k = 0, 1, 2, \dots, \overline{m-1}). \tag{6}$$

Without affecting the generality of f_{8m}, we take $u_i = a_{0m} = 1$, and write

$$f_{8m} = y^m + \sum_{r=1}^{m} (a_{rm-r}x^r + a_{r-1\,m-r}x^{r-1} + \dots + a_{0m-r})\, y^{m-r},$$

$$\sigma_i = y^2 + (t_i x + w_i)\, y + (s_i x^2 + v_i x + r_i).$$

Hence by § 2

$$\Phi^{(m)}_{-(t_i x + w_i)} = 0, \quad \Psi^{(m)}_{-(s_i x^2 + v_i x + r_i)} = 0,$$

wherein the usual coefficients a_r have been replaced by

$$a_{rm-r}x^r + a_{r-1\,m-r}x^{r-1} + \dots + a_{0m-r} \qquad (r = 1, 2, \dots, m).$$

It follows that the coefficients t_i and s_i are respectively roots of the equations

$$\Phi^{(m)}_{-t} = t^{(m)} - \phi_1 t^{(m)-1} + \phi_2 t^{(m)-2} - \dots + (-1)^{(m)} \phi_{(m)} = 0, \tag{7}$$

$$\Psi^{(m)}_{-s} = s^{(m)} - \psi_1 s^{(m)-1} + \psi_2 s^{(m)-2} - \dots + (-1)^{(m)} \psi_{(m)} = 0, \tag{8}$$

in which the usual coefficients a_r (see § 2) are now replaced by a_{rm-r}. Moreover the coefficients ϕ, ψ are rational in the elementary symmetric functions a of the groups of quantities

$$s_i,\ t_i,\ u_i,\ v_i,\ w_i,\ r_i \qquad (i = 1, 2, \ldots, g).$$

When these symmetric functions a are expressed in terms of the s_i, t_i, ..., and s, t are replaced by s_j, t_j, respectively, for any value of j, Φ_{-t} and Ψ_{-s} vanish identically. When so expressed, we write them $I(s_i, \ldots)$, $J(s_i, \ldots)$ respectively. Hence

$$\Phi(a, t_j) = I(s_i, \ldots;\ t_j) \equiv 0,$$
$$\Psi(a, s_j) = J(s_i, \ldots;\ s_j) \equiv 0.$$

Now in I let s_i, t_i be replaced respectively by $s_i + \lambda v_i$, $t_i + \lambda w_i$ (t by $t + \lambda w$). Then expand by Taylor's theorem

$$I(s_i + \lambda v_i,\ t_i + \lambda w_i,\ \ldots;\ t + \lambda w),$$

and by equating coefficients we get

$$\sum \frac{\partial I}{\partial s_1} v_1 + \sum \frac{\partial I}{\partial t_1} w_1 + \frac{\partial I}{\partial t} w = \sum \frac{\partial \Phi}{\partial s_1} v_1 + \sum \frac{\partial \Phi}{\partial t_1} w_1 + \frac{\partial \Phi}{\partial t} w.$$

But

$$\sum \frac{\partial \Phi}{\partial s_1} v_1 + \sum \frac{\partial \Phi}{\partial t_1} w_1 = \sum \frac{\partial \Phi}{\partial a_{m-kk}} \Big(\sum \frac{\partial a_{m-kk}}{\partial s_1} v_1 + \sum \frac{\partial a_{m-kk}}{\partial t_1} w_1 \Big)$$
$$= \sum \frac{\partial \Phi}{\partial a_{m-kk}} a_{m-k-1k} \quad \text{(from (6))}.$$

Therefore we have for the determination of w_j ($j = 1, 2, \ldots, g$)

$$w_j \frac{\partial \Phi_{-t_j}}{\partial t_j} + \sum_k \frac{\partial \Phi_{-t_j}}{\partial a_{m-kk}} a_{m-k-1k} = 0. \tag{9}$$

By a similar expansion of $\psi(a, s)$ we get for the determination of v_j

$$v_j \frac{\partial \Psi_{-s_j}}{\partial s_j} + \sum_k \frac{\partial \Psi_{-s_j}}{\partial a_{m-kk}} a_{m-k-1k} = 0. \tag{10}$$

Thus the factors σ_i *of* f_{3m} *are completely determined from* (7), (8), (9), (10).

Next let the quaternary form

$$f_{4m}(x, y, z, 1) = y^m + \sum_{r=1}^{m} \Big[\sum_{s=0}^{r} (a_{sm-rr-s} x^s + a_{s-1m-rr-s} x^{s-1} + \cdots + a_{0m-rr-s}) z^{r-s} \Big] y^{m-r}$$

equal the product of the quadratic forms

$$\tau_i = y^2 + (o_i x + s_i z + u_i) y + (n_i x^2 + q_i xz + r_i z^2 + t_i x + v_i z + w_i)$$
$$[i = 1, 2, \ldots, g = \tfrac{1}{2} m].$$

Then we have, as in (6),

$$\left.\begin{aligned}
&\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial n_1}t_1+\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial o_1}u_1+\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial q_1}v_1=a_{s-1\,m-r\,r-s}\ \ (s>0),\\
&\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial o_1}t_1+\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial p_1}u_1+\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial s_1}v_1=a_{s\,m-r-1\,r-s}\ \ (r<m),\\
&\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial q_1}t_1+\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial s_1}u_1+\sum\frac{\partial a_{s\,m-r\,r-s}}{\partial r_1}v_1=a_{s\,m-r\,r-s-1}\ \ (s<r),
\end{aligned}\right\}\quad(11)$$

and also as in (7), (8) that the coefficients o_i and n_i are roots of the respective equations

$$\Phi_{-o}^{(m)}=o^{(m)}-\phi_1\,o^{(m)-1}+\phi_2\,o^{(m)-2}-\ldots\ldots+(-1)^{(m)}\phi_{(m)}=0,\qquad(12)$$

$$\Psi_{-n}^{(m)}=n^{(m)}-\psi_1\,n^{(m)-1}+\psi_2\,n^{(m)-2}-\ldots\ldots+(-1)^{(m)}\psi_{(m)}=0.\qquad(13)$$

In these the usual coefficients a_r [§ 2] are replaced by $a_{r\,m-r\,0}$ $[r=1,2,\ldots,m]$. Now when the coefficients $a_{r\,m-r\,0}$ in these functions (12), (13) are expressed as elementary symmetric functions of the groups of variables

$$n_i,\ o_i,\ p_i,\ q_i,\ r_i,\ s_i,\ t_i,\ u_i,\ v_i,\ w_i\quad[i=1,2,\ldots,g],$$

we write them as

$$I(n_i,\ o_i,\ \ldots,\ w_i;\ o),\quad J(n_i,\ o_i,\ \ldots,\ w_i;\ n)$$

respectively. From the expansion of

$$I(n_i+\lambda t_i,\ o_i+\lambda u_i,\ p_i,\ q_i+\lambda v_i,\ r_i,\ \ldots,\ w_i;\ o+\lambda u),$$

we get

$$\begin{aligned}
&\sum\frac{\partial I}{\partial n_1}t_1+\sum\frac{\partial I}{\partial o_1}u_1+\sum\frac{\partial I}{\partial q_1}v_1+\frac{\partial I}{\partial q}v\\
&\quad=\sum\frac{\partial\Phi}{\partial a_{m-k\,k\,0}}\left[\sum\frac{\partial a_{m-k\,k\,0}}{\partial n_1}t_1+\sum\frac{\partial a_{m-k\,k\,0}}{\partial o_1}u_1+\sum\frac{\partial a_{m-k\,k\,0}}{\partial q_1}v_1\right]+\frac{\partial\Phi}{\partial o}u\\
&\quad=\sum\frac{\partial\Phi}{\partial a_{m-k\,k\,0}}\,a_{m-k-1\,k\,0}+\frac{\partial\Phi}{\partial o}u\qquad\text{(from (11))}.
\end{aligned}$$

Hence we have for the determination of u_j

$$u_j\frac{\partial\Phi_{-o_j}}{\partial o_j}+\sum_k\frac{\partial\Phi_{-o_j}}{\partial a_{m-k\,k\,0}}\,a_{m-k-1\,k\,0}=0.\qquad(14)$$

Similarly from the expansion of

$$J(n_i+\lambda t_i,\ o_i+\lambda u_i,\ p_i,\ q_i+\lambda v_i,\ r_i,\ \ldots,\ w_i;\ n+\lambda t)$$

and the first equation of (11) there results

$$t_j \frac{\partial \Psi_{-n_j}}{\partial n_j} + \sum_k \frac{\partial \Psi_{-n_j}}{\partial a_{m-k\,k\,0}} a_{m-k-1\,k\,0} = 0. \tag{15}$$

Again the coefficients r_i are roots of the equation

$$\Psi^{(m)}_{-r} = r^{(m)} - \psi_1 r^{(m)-1} + \psi_2 r^{(m)-2} - \dots + (-1)^{(m)} \psi_{(m)} = 0, \tag{16}$$

wherein the usual a_r [§ 2] are replaced by $a_{0\,m-r\,r}$ $[r = 1, \dots, m]$. Hence, from the third equation of (11) we have, by expanding

$$\Psi(a) = J(n_i,\ o_i,\ p_i,\ q_i + \lambda t_i,\ r_i + \lambda v_i;\ s_i + \lambda u_i,\ t_i,\ \dots,\ w_i;\ r + \lambda v),$$

$$v_j \frac{\partial \Psi_{-r_j}}{\partial r_j} + \sum_k \frac{\partial \Psi_{-r_j}}{\partial a_{0\,m-k\,k}} a_{0\,m-k\,k-1} = 0, \tag{17}$$

a formula for the determination of v_j.

Next let us write the form f_{4m} as a quantic in the three variables x, y, w, taking $z = 1$. That is, let

$$f_{4m}(x, y, 1, w) = \prod_{i=1}^{g} \tau_i(x, y, 1, w). \tag{18}$$

Then we have

$$f_{4m}(x, y, 1, w) = y^m + \sum_{r=1}^{m} \Big[\sum_{s=0}^{r} (a_{s\,m-r\,0} x^s + a_{s-1\,m-r\,1} x^{s-1} + \dots + a_{0\,m-r\,s}) w^{r-s}\Big] y^{m-r},$$

$$\tau_i = y^2 + (o_i x + u_i w + s_i) y + (n_i x^2 + t_i x w + w_i w^2 + q_i x + v_i w + r_i)$$
$$(i = 1, \dots, g).$$

It follows that the u_j determined by (14) satisfy the equation

$$\Phi^{(m)}_{-u} = u^{(m)} - \phi_1 u^{(m)-1} + \phi_2 u^{(m)-2} - \dots + (-1)^{(m)} \phi_{(m)} = 0, \tag{19}$$

wherein the usual a_r [§ 2] are replaced by $a_{0\,m-r\,0}$ $[r = 1, \dots, m]$, and that we again get equation (13) for the determination of the n_j. Also, on account of the relation (18), we have, as in (6) and (11),

$$\Big(\sum \frac{\partial}{\partial t_1} q_1 + \sum \frac{\partial}{\partial u_1} s_1 + \sum \frac{\partial}{\partial w_1} v_1\Big) a_{s\,m-r\,0} = a_{s\,m-r\,1} \quad \begin{bmatrix} s = 0, 1, \dots, r \\ r = 1, 2, \dots, m \end{bmatrix},$$

$$\Big(\sum \frac{\partial}{\partial n_1} q_1 + \sum \frac{\partial}{\partial o_1} s_1 + \sum \frac{\partial}{\partial t_1} v_1\Big) a_{s\,m-r\,0} = a_{s-1\,m-r\,1} \quad (s > 0).$$

Hence, by applying Taylor's expansion as above we obtain for the determination of s_j, q_j respectively

$$s_j \frac{\partial \Phi_{-u_j}}{\partial u_j} + \sum_k \frac{\partial \Phi_{-u_j}}{\partial a_{0\,m-k\,0}} a_{0\,m-k\,1} = 0, \tag{20}$$

$$q_j \frac{\partial \Psi_{-n_j}}{\partial n_j} + \sum_k \frac{\partial \Psi_{-n_j}}{\partial a_{k\,m-k\,0}} a_{k-1\,m-k\,1} = 0. \tag{21}$$

The coefficient w_j satisfies an equation analogous to (13). *Hence equations* (12), (13), (14), (15), (16), (17), (20), (21) *completely determine the component factors* τ *of* f_{4m}.

§ 5. *Cubic Factors.*

If the ternary form

$$f_{3m} = a_{0m}\, y^m + \sum_{r=1}^{m} (a_{r\,m-r}\, x^r + a_{r-1\,m-r}\, x^{r-1} + \dots + a_{0\,m-r})\, y^{m-r}$$

can be split into $g = \frac{1}{3} m$ cubic factors

$$\rho_i = q_i y^3 + (p_i x + t_i) y^2 + (o_i x^2 + s_i x + v_i) y + (n_i x^3 + r_i x^2 + u_i x + w_i) \qquad (i = 1, \dots, g),$$

its coefficients are connected with the coefficients of these cubics by the relations

$$\left.\begin{aligned} \left(\sum \frac{\partial}{\partial n_1} r_1 + \sum \frac{\partial}{\partial o_1} s_1 + \sum \frac{\partial}{\partial p_1} t_1\right) a_{m-r\,r} &= a_{m-r-1\,r}, \\ \left(\sum \frac{\partial}{\partial o_1} r_1 + \sum \frac{\partial}{\partial p_1} s_1 + \sum \frac{\partial}{\partial q_1} t_1\right) a_{m-r\,r} &= a_{m-r\,r-1}, \\ \left(\sum \frac{\partial}{\partial w_i} u_i + \sum \frac{\partial}{\partial v_i} s_i + \sum \frac{\partial}{\partial t_i} p_i\right) a_{0\,m-r} &= a_{1\,m-r}. \end{aligned}\right\} \qquad (22)$$

Also the coefficients p, o, n satisfy the following equations respectively:

$$\Phi_{1,-p}^{C_3^m} = p^{C_3^m} - \phi_{11} p^{C_3^m-1} + \phi_{12} p^{C_3^m-2} - \dots + (-1)^{C_3^m} \phi_{1\,C_3^m} = 0, \qquad (23)$$

$$\Phi_{2,-o}^{C} = o^{C_3^m} - \phi_{21} o^{C_3^m-1} + \phi_{22} o^{C_3^m-2} - \dots + (-1)^{C_3^m} \phi_{2\,C_3^m} = 0, \qquad (24)$$

$$\Phi_{3,-n}^{C_3^m} = n^{C_3^m} - \phi_{31} n^{C_3^m-1} + \phi_{32} n^{C_3^m-2} - \dots + (-1)^{C_3^m} \phi_{3\,C_3^m} = 0, \qquad (25)$$

wherein the coefficient a_r of § 2 are now replaced by $a_{r\,m-r}$ $[r = 1, 2, \dots, m]$. Hence by Taylor's theorem and the first equation of (22) we get

$$t_j \frac{\partial \Phi_{1,-p_j}}{\partial p_j} + \sum_k \frac{\partial \Phi_{1,-p_j}}{\partial a_{m-k\,k}}\, a_{m-k-1\,k} = 0, \qquad (26)$$

$$s_j \frac{\partial \Phi_{2,-o_j}}{\partial o_j} + \sum_k \frac{\partial \Phi_{2,-o_j}}{\partial a_{m-k\,k}}\, a_{m-k-1\,k} = 0, \qquad (27)$$

$$r_j \frac{\partial \Phi_{3,-n_j}}{\partial n_j} + \sum_k \frac{\partial \Phi_{3,-n_j}}{\partial a_{m-k\,k}}\, a_{m-k-1\,k} = 0. \qquad (28)$$

Again writing f_{3m} as a homogeneous form in x, y, z and then taking $x = 1$, we see that v_i and w_i satisfy the respective equations

$$\Phi_{2,-v}^{C_3^m} = 0, \quad \Phi_{3,-w}^{C_3^m} = 0, \qquad (29)$$

wherein the a_r of § 2 are replaced by $a_{0\,m-r}$ $[r = 1, 2, \ldots, m]$. Hence by Taylor's theorem and the third equation of (22) we get for the determination of the remaining coefficients u_j

$$u_j \frac{\partial \Phi_{3,-w_j}}{\partial w_j} + \sum_k \frac{\partial \Phi_{3,-w_j}}{\partial a_{0\,m-k}} a_{1\,m-k} = 0. \tag{30}$$

The cubic factors of f_{3m} are completely determined by formulas (23)....(30).

§ 6. *Asymptotic Forms of a Given Form.*

Using a well-known notation let us write

$$f_{3m}(x, y, z) = u_m + u_{m-1} z + u_{m-2} z^2 + \ldots + u_0 z^m = \sum_{i=0}^{m} u_{m-i} z^i.$$

Let the expressions

$$v_{1\tau_1},\ v_{2\tau_2},\ \ldots,\ v_{\nu\tau_\nu} \quad (\Sigma \tau_i = m)$$

be any set of real factors of u_m which are algebraically prime to each other. Let u_{m-1}/u_m be resolved into partial fractions, and suppose that the numerators corresponding respectively to the above denominators are

$$v_{1\,\tau_1-1},\ v_{2\,\tau_2-1},\ \ldots,\ v_{\nu\,\tau_\nu-1}.$$

Then

$$f_{3m} = \zeta_{1\tau_1} \zeta_{2\tau_2} \ldots \zeta_{\nu\tau_\nu} + z^2 \mu_{m-2} = \prod_i \zeta_{i\tau_i} + z^2 \mu_{m-2}, \tag{31}$$

where

$$\zeta_{i\tau_i} = v_{i\tau_i} + v_{i\,\tau_i-1} z + R_{i\,\tau_i-2} z^2,$$

the $(\tau_i - 2)$-ic R being arbitrary. The forms $\zeta_{i\tau_i}$ will be called asymptotic forms of f_{3m}. The $(m-2)$-ic μ_{m-2} will be called the satellite form of f_{3m}. Both μ_{m-2} and the satellite forms ζ are affected as to form by the choice of $R_{i\,\tau_i-2}$ $(i = 1, 2, \ldots, \nu)$.

Now if the $v_{i\tau_i}$ are the real factors of u_m of lowest degree prime to each other, assuming for the present that there are no repeated quadratic factors and no linear factor of multiplicity exceeding two, we have $\tau_i = 1$ or 2. We first show that *if by small variations of its coefficients a form f_{3m} decomposes into a form of degree $(m-k)$ and a factor of degree k $(=1, 2)$, then this factor is identical with one of f_{3m}'s asymptotic forms ζ_{i_k}.*

Take $R_{i\,\tau_i-2} = 0$, and place $z = 1$ in (31). Then we have

$$\prod_{i=1}^{\nu} \zeta_{i\tau_i} = u_m + u_{m-1} + g_{m-2} = h_{3m}(x, y, 1).$$

This form h_{3m} is thus degenerate and formulas (7), (8), (9), (10), § 3,* apply for the determination of its factors $\zeta_{i\tau_i}$. And since these formulas involve only the coefficients of u_m, u_{m-1}, which are common to h_{3m} and f_{3m}, it follows that the asymptotic forms of f_{3m} are formally identical with its component forms σ_i when it is degenerate, which proves the statement.*

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* To these add Junker's formulas for the determination of linear factors, *Mathem. Annalen*, Bd. 45, S. 39.

Let us investigate the analogous properties of quaternary forms. Let [§ 4]

$$f_{4m}(x, y, z, w) = \sum_{i=0}^{m} u_{m-i}\, w^i.$$

Assume that the ternary form u_m may be decomposed into linear factors

$$u_m = \prod_{k=1}^{m} (x + r_k y + s_k z).$$

Then it is not difficult to show that the necessary and sufficient condition that it be possible to resolve u_{m-1}/u_m into the partial fractions

$$\frac{u_{m-1}}{u_m} = \sum_{k=1}^{m} \frac{\alpha_k}{x + r_k y + s_k z}$$

is the vanishing of each determinant of the matrix

$$M_{m0}^{(m)} = \left|\begin{matrix} a_{m-100} & a_{m-210} & \cdots a_{0m-10}, & a_{m-201} & a_{m-311} & \cdots a_{0m-21}, & \cdots a_{00m-1} \\ \chi_{00}^{1} & \chi_{10}^{1} & \cdots \chi_{m-10}^{1}, & \chi_{01}^{1} & \chi_{11}^{1} & \cdots \chi_{m-21}^{1}, & \cdots \chi_{0m-1}^{1} \\ \chi_{00}^{2} & \chi_{10}^{2} & \cdots \chi_{m-10}^{2}, & \chi_{01}^{2} & \chi_{11}^{2} & \cdots \chi_{m-21}^{2}, & \cdots \chi_{0m-1}^{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \chi_{00}^{\kappa} & \chi_{10}^{\kappa} & \cdots \chi_{m-10}^{\kappa}, & \chi_{01}^{\kappa} & \chi_{11}^{\kappa} & \cdots \chi_{m-21}^{\kappa}, & \cdots \chi_{0m-1}^{\kappa} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \chi_{00}^{\lambda} & \chi_{10}^{\lambda} & \cdots \chi_{m-10}^{\lambda}, & \chi_{01}^{\lambda} & \chi_{11}^{\lambda} & \cdots \chi_{m-21}^{\lambda}, & \cdots \chi_{0m-1}^{\lambda} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \chi_{00}^{m} & \chi_{10}^{m} & \cdots \chi_{m-10}^{m}, & \chi_{01}^{m} & \chi_{11}^{m} & \cdots \chi_{m-21}^{m}, & \cdots \chi_{0m-1}^{m} \end{matrix}\right|, \qquad (32)$$

---

* An interesting theorem on partial fractions results from the fact that the $\zeta_{i\,r_i}$ may be determined both by the partial fraction method and the differential formulas of the preceding sections:

Let

$$\frac{u_{m-1}}{u_m} = \frac{\sum\limits_{i=0}^{m-1} a_{m-1-i\,i}\, x^{m-1-i}\, y^i}{\sum\limits_{j=0}^{m} a_{m-j\,j}\, x^{m-j}\, y^j} = \frac{u_{m-1}}{\prod\limits_{h} (x - r_h y) \prod\limits_{k} (x^2 + \xi_k x y + \eta_k y^2)}.$$

When this is reduced to partial fractions, the numerator of that fraction which has $x^2 + \xi_k xy + \eta_k y^2$ for a denominator may be obtained by differentiation. It is

$$-\frac{\delta \Phi_{-\xi_k}^{(m)}}{\delta' \Phi_{-\xi_k}^{(m)}}\, x - \frac{\delta \Psi_{-\eta_k}^{(m)}}{\delta' \Psi_{-\eta_k}^{(m)}}\, y = v_{k,1},$$

where $\delta' = \dfrac{\partial}{\partial x}$, and δ is the Aronhold operator

$$\delta = a_{m-10} \frac{\partial}{\partial a_{m-11}} + a_{m-21} \frac{\partial}{\partial a_{m-22}} + \cdots + a_{0\,m-1} \frac{\partial}{\partial a_{0m}}.$$

where $\chi^{\lambda}_{\mu\nu}$ is the elementary symmetric function of the $m-1$ groups of variables $P_k(r_k, s_k)$ $[k = 1, 2, \ldots, \lambda-1, \lambda+1, \ldots, m]$, each term of which involves μ r's and ν s's.* If in this matrix the two groups $P_\kappa(r_\kappa, s_\kappa)$, $P_\lambda(r_\lambda, s_\lambda)$ are interchanged, the only effect will be to interchange the κth and λth rows of the matrix. Consequently: *Every determinant of the matrix $M^{(m)}_{m0}$ is an alternating function of the variables r, s.* More generally, the condition that u_{m-1}/u_m be resolvable as follows:

$$\frac{u_{m-1}}{u_m} = \sum_{i=1}^{k} \frac{a_i}{x + r_i y + s_i z} + \sum_{j=1}^{l} \frac{\beta_j x + \gamma_j y + \delta_j z}{x^2 + t_j xy + u_j y^2 + v_j xz + w_j yz + x_j z^2},$$

is the vanishing of a determinate matrix $M^{(m)}_{kl}$. This is a matrix of $\frac{1}{2} m(m-1)$ columns and $\frac{1}{2}(3m-k+2)$ rows, the elements being functions of the coefficients of u_{m-1} and of the factors of u_m.

When u_m can be thus resolved into k linear and $l = \frac{1}{2}(m-k)$ quadratic factors, and $M^{(m)}_{kl} = 0$, then f_{4m} can be thrown into the form

$$f_{4m} = \zeta_{1r_1}\zeta_{2r_2}\cdots\zeta_{\nu r_\nu} + w^2\mu_{m-2} = 0, \tag{33}$$

where

$$\zeta_{ir_i} = v_{ir_i} + v_{ir_i-1} w + R_{r_i-2} w^2,$$

the v_{ir_i} being the factors of u_m, while v_{ir_i-1} is the numerator of the partial fraction having v_{ir_i} for a denominator. As before, R is arbitrary. We now have, placing $R_{r_i-2} = 0$, $w = 1$,

$$\prod_{i=1}^{\nu} \zeta_{ir_i} = u_m + u_{m-1} + g_{m-2} = h_{4m}(x, y, z, 1).$$

Hence [§ 4, (12) to (21)], to summarize these results: *The necessary and sufficient conditions that a quaternary form $\sum_{i=0}^{m} u_{m-i} w^i$ may possess a set of asymptotic forms of degree ν $(=1, 2)$ are that u_m may be factorable into linear and quadratic factors, and that $M^{(m)}_{kl} = 0$. As the form is altered by small variations of its coefficients, the necessary condition that it degenerate into k linear and l quadratic factors is $M^{(m)}_{kl} = 0$. And in its degenerate form the component factors coincide with the asymptotic forms ζ.*

§ 7. *Summary of Geometrical Results.*

The effect of the differential formulas of § 3, when applied to any form f, is to put in evidence the asymptotic forms of that form. That is, by these formulas we reduce the given form to the type (31) or (33). If the form proves to be degenerate, μ_{m-2} vanishes identically, so that our formulas determine the component factors. The necessary and sufficient conditions that the form f represent a penultimate curve (surface), penultimate to a set of lines and conics (planes and conicoids), are that the coefficients of the terms of the satellite be all infinitesimal. *If by small variations of its coefficients the locus $f=0$ degenerates into a set of lines and conics (planes and conicoids), these lines and conics (planes and conicoids) are the curves (surfaces) which the asymptotic curves (surfaces) of f, $\zeta_{i r_i} = 0$ $[i = 1, 2, \ldots, \nu]$, approach as limiting curves (surfaces) during that variation.* The functional member of the equation of any penultimate curve (surface) can be expressed as the sum of a product of a set of asymptotic forms and a satellite function having infinitesimal coefficients.

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* Thus if $m = 3$, $\chi^{2}_{11} = r_1 s_3 + r_3 s_1$ (see Junker, *loc. cit.*, p. 4). Also $\chi^{\lambda}_{00} = 1$ $[\lambda = 1, 2, \ldots, m]$.

PART II.

ON PENULTIMATE CURVES AND SURFACES.

§ 1. *Conditions for Degeneracy.*

It has been proved that the coefficients of a degenerate ternary form f_{3m} are connected by exactly

$$s = \tfrac{1}{2}(m^2 - \Sigma r_i)$$

independent relations, $r_1, r_2, \ldots$ being the respective orders of the component factors. If $r_i = 1$, $s = \frac{1}{2} m(m-1)$. *Hence, since the number of coefficients in the satellite μ_{m-2} is precisely $\frac{1}{2} m(m-1)$, these equated to zero are the necessary and sufficient conditions that f_{3m} degenerate into m straight lines.* We proceed to determine the satellite μ_{m-2} for this case $r_i = 1$, of the forms f_{33}, f_{34}.

Let

$$\begin{aligned} f_{33}(x_1, x_2, 1) &= a_x^3 + b_x^2 + c_x + d, \\ d &= d_0, \\ c_x &= c_0 x_1 + c_1 x_2, \\ b_x^2 &= b_0 x_1^2 + b_1 x_1 x_2 + b_2 x_2^2, \\ a_x^3 &= a_0 x_1^3 + a_1 x_1^2 x_2 + a_2 x_1 x_2^2 + a_3 x_2^3 \\ &= (x_1 - r_1 x_2)(x_1 - r_2 x_2)(x_1 - r_3 x_2) \quad [a_0 = 1]. \end{aligned}$$

By the method of § 5 we get the linear satellite of f_{33},

$$\begin{aligned} D\mu_1 = {} & [S b_{r_1}^2 b_{r_2}^2 (r_3 - r_1)(r_2 - r_3) - c_0 D] x_1 + [S b_{r_1}^2 b_{r_2}^2 (r_3 - r_1)(r_2 - r_3) r_3 - c_1 D] x_2 \\ & - [b_{r_1}^2 b_{r_2}^2 b_{r_3}^2 + d_0 D] = 0 = H_1 x_1 + H_2 x_2 + H_3 \text{ (say)}, \end{aligned} \tag{34}$$

where D is the discriminant of the form a_x^3, and the summation S includes the terms obtained from the given term by the permutations of the cyclic group $S = \{1, (r_1 r_2 r_3), (r_1 r_3 r_2)\}$. The coefficients H_i are symmetric in the roots r_1, r_2, r_3, and so we can express them in terms of the coefficients $a_j\ [j = 1, 2, 3]$. When this is done, the satellite $D\mu_1$ takes the form given on the following page. *The three independent conditions that f_{33} degenerate into three straight lines are accordingly*

$$H_1 = 0, \quad H_2 = 0, \quad H_3 = 0.$$

H_1	$-H_2$	$-H_3$
$4\,a_1 a_2 a_3 b_0^2$	$a_2^2 a_3 b_0^2$	$a_3^2 b_0^3$
$-\,a_2^3 b_0^3$	$-\,3\,a_1 a_3^2 b_0^3$	$+\,a_1 a_3 b_0 b_1^2$
$-\,9\,a_3^2 b_0^3$	$+\,a_1^2 a_3 b_1^2$	$+\,a_1^2 b_0 b_2^2$
$+\,3\,a_1 a_3 b_1^3$	$-\,3\,a_2 a_3 b_1^3$	$-\,2\,a_2 b_0 b_2^2$
$-\,a_2^2 b_1^2$	$+\,a_1^3 b_2^2$	$-\,a_2 a_3 b_0^2 b_1$
$+\,3\,a_2 b_2^3$	$-\,4\,a_1 a_2 b_2^2$	$+\,a_2^2 b_0^2 b_2$
$-\,a_1^2 b_2^3$	$+\,9\,a_3 b_2^2$	$-\,2\,a_1 a_3 b_0^2 b_2$
$+\,6\,a_1 a_3 b_0 b_2$	$+\,9\,a_3^2 b_0 b_1$	$+\,3\,a_3 b_0 b_1 b_2$
$-\,2\,a_2^2 b_0 b_2$	$-\,a_1 a_2 a_3 b_0 b_1$	$-\,a_1 a_2 b_0 b_1 b_2$
$+\,a_1 a_2^2 b_0 b_1$	$+\,2\,a_1^2 a_3 b_0 b_2$	$-\,a_3 b_1^3$
$+\,3\,a_2 a_3 b_0 b_1$	$-\,6\,a_2 a_3 b_0 b_2$	$-\,a_1 b_1 b_2^2$
$-\,4\,a_1^2 a_3 b_0 b_1$	$+\,4\,a_2^2 b_1 b_2$	$+\,a_2 b_1^2 b_2$
$+\,a_1 a_2 b_1 b_2$	$-\,3\,a_1 a_3 b_1 b_2$	$+\,b_2^3$
$-\,9\,a_3 b_1 b_2$	$-\,a_1^2 a_2 b_1 b_2$	$+\,a_1^2 a_2^2 d_0$
$-\,a_1^2 a_2^2 c_0$	$+\,a_1^2 a_2^2 c_1$	$+\,18\,a_1 a_2 a_3 d_0$
$-\,18\,a_1 a_2 a_3 c_0$	$+\,18\,a_1 a_2 a_3 c_1$	$-\,4\,a_2^3 d_0$
$+\,4\,a_2^3 c_0$	$-\,4\,a_2^3 c_1$	$-\,4\,a_1^3 a_3 d_0$
$+\,4\,a_1^3 a_3 c_0$	$-\,4\,a_1^3 a_3 c_1$	$-\,27\,a_3^2 d_0$
$+\,27\,a_3^2 c_0$	$-\,27\,a_3^2 c_1$	

By the same methods we derive the satellite conic of f_{34}. Let

$$f_{34}(x_1, x_2, 1) = a_x^4 + b_x^3 + c_x^2 + d_x + e \quad (a_0 = 1),$$

and

$$a_x^4 = (x_1 - r_1 x_2)(x_1 - r_2 x_2)(x_1 - r_3 x_2)(x_1 - r_4 x_2).$$

Then we get

$$\left.\begin{aligned} D\mu_2 = {} & [S_1 b^3_{r_1} b^3_{r_2} (r_1 - r_3)(r_1 - r_4)(r_2 - r_3)(r_2 - r_4)(r_3 - r_4)^2 + c_0 D]\, x_1^2 \\ & + [S_1 b^3_{r_1} b^3_{r_2} (r_3 - r_1)(r_1 - r_4)(r_2 - r_3)(r_2 - r_4)(r_3 - r_4)^2 (r_3 + r_4) + c_1 D]\, x_1 x_2 \\ & + [S_1 b^3_{r_1} b^3_{r_2} (r_1 - r_3)(r_1 - r_4)(r_2 - r_3)(r_2 - r_4)(r_3 - r_4)^2 r_3 r_4 + c_2 D]\, x_2^2 \\ & + [S_2 b^3_{r_1} b^3_{r_2} b^3_{r_3} (r_1 - r_4)(r_4 - r_2)(r_4 - r_3) + d_0 D]\, x_1 \\ & + [S_2 b^3_{r_1} b^3_{r_2} b^3_{r_3} (r_4 - r_1)(r_4 - r_2)(r_4 - r_3)\, r_4 + d_1 D]\, x_2 \\ & + [-b^3_{r_1} b^3_{r_2} b^3_{r_3} b^3_{r_4} + e_0 D] = 0 \\ = {} & I_1 x_1^2 + I_2 x_1 x_2 + I_3 x_2^2 + I_4 x_1 + I_5 x_2 + I_6 = 0. \end{aligned}\right\} \quad (35)$$

Each summation S_1 includes the six terms obtained from the given leading term by operating the three powers of the permutation $(r_1 r_2 r_3)$, and these same three powers preceded by the transposition $(r_1 r_4)$. Each summation S_2 includes all terms obtained from the given leading term by the permutations of the cyclic group $S_2 = \{1, (r_1 r_2 r_3 r_4), (r_1 r_3)(r_2 r_4), (r_1 r_4 r_3 r_2)\}$. The coefficients I_j are symmetric in the r's. Expressing them in terms of the coefficients of a_x^4 we have the satellite* conic. *The six independent conditions that f_{34} degenerate into four straight lines are* $I_j = 0$ $[j = 1, 2, 3, 4, 5, 6]$.

It is now evident that the degree (in the coefficients of f_{3m}) of the satellite $D\mu_{m-2}$ of the form f_{3m} is just one greater than the degree of the discriminant of

$$a_x^m = (1, a_1, a_2, \ldots, a_m \between x_1, x_2)^m.$$

Hence the degree† *of each of the* $\frac{1}{2} m(m-1)$ *conditions that the general ternary* m-ic $f_{3m} = 0$ *degenerate into* m *straight lines is exactly* $2m - 1$.

We derive next the conditions that $f_{33} = 0$ degenerate into a straight line and a conic.

Let

$$a_x^3 = (x_1 - r x_2)\,[x_1^2 + (r + a_1)\, x_1 x_2 + (r^2 + a_1 r + a_2)\, x_2^2],$$

where r is real. For the satellite line we get, after multiplying by $(3r^2 + 2a_1 r + a_2)^2$‡,

$$J_1' x_1 + J_2' x_2 + J_3' = 0,$$

$$\begin{aligned} J_1' &= (b_0 r^2 + b_1 r + b_2)\,[2 b_0 r^3 + (2 a_1 b_0 - b_1)\, r + (a_2 b_0 - b_2)] \\ &\qquad - (c_0 - R_0)(3r^2 + 2a_1 r + a_2)^2, \\ J_2' &= (b_0 r^2 + b_1 r + b_2)\,[b_1 r^2 + (a_1 b_1 - 2 b_2)\, r + (a_2 b_1 - a_1 b_2 - a_3 b_0)] \\ &\qquad - (c_1 + r R_0)(3r^2 + 2a_1 r + a_2)^2, \\ J_3' &= (3r^2 + 2a_1 r + a_2)\,[(b_0 r^2 + b_1 r + b_2)\, R_0 - (3r^2 + 2a_1 r + a_2)\, d_0]. \end{aligned}$$

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* The satellite of the ternary quartic μ_2 has also been computed. The six coefficients I_j are each about equal in size to the eighth-degree invariant (J_8) of the quintic. These will be published later.

† An important application is the result that a pencil of curves $f_{3m} + \lambda \phi_{3m}$ contains at most $2m - 1$ degenerate members consisting of m straight lines. This maximum can be reached. Cf. Hadamard, *loc. cit.*

‡ This cannot vanish, since a_x has no multiple roots, by hypothesis.

Elimination of R_0 from J_i' $[i = 1, 2, 3]$ gives the two conditions for the degeneracy of f_{33} into a straight line and a conic. There are four cases, corresponding to the four ways in which $J_3' = 0$ can be satisfied.

CASE 1. $b_r^2 = b_0 r^2 + b_1 r + b_2 = 0$, $d_0 = 0$, $R_0 = 0$. The conditions are $c_0 = c_1 = 0$, and the vanishing of the resultant of b_r^2, a_r^3. This is

$$J_1 = \begin{vmatrix} b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \\ a_1 b_0 - b_1 & a_2 b_0 - b_2 & a_3 b_0 & 0 \\ 0 & a_1 b_0 - b_1 & a_2 b_0 - b_2 & a_3 b_0 \end{vmatrix} = 0.$$

CASE 2. $d_0 = 0$, $R_0 = c_0$. Here $b_r = 0$, $c_r = 0$, so that the desired conditions are $J_1 = 0$, $b_0 c_1^2 - b_1 c_1 c_0 + b_2 c_0^2 = 0$.

CASE 3. $d_0 = 0$, $R_0 = 0$, $b_r^2 \neq 0$. The two conditions are the vanishing of the resultants of a_r^3 and the two equations (in r) $J_1' = 0$, $J_2' = 0$, viz.:

$$J_2 = \begin{vmatrix} 1 & a_1 & a_2 & a_3 & 0 \\ 0 & 1 & a_1 & a_2 & a_3 \\ \begin{array}{l} a_1 b_0 b_1 \\ - b_1^2 \\ + b_0 b_2 \\ - a_2 b_0^2 \\ + 3 a_2 c_0 \\ - a_1^2 c_0 \end{array} & \begin{array}{l} 2 a_1 b_0 b_2 \\ - 2 a_3 b_0^2 \\ - 2 b_1 b_2 \\ + 9 a_3 c_0 \\ - a_1 a_2 c_0 \end{array} & \begin{array}{l} a_2 b_0 b_2 \\ - a_3 b_0 b_1 \\ - b_2^2 \\ + 3 a_1 a_3 c_0 \\ - a_2^2 c_0 \end{array} & 0 & 0 \\ 0 & \begin{array}{l} a_1 b_0 b_1 \\ - b_1^2 \\ + b_0 b_2 \\ - a_2 b_0^2 \\ + 3 a_2 c_0 \\ - a_1^2 c_0 \end{array} & \begin{array}{l} 2 a_1 b_0 b_2 \\ - 2 a_3 b_0^2 \\ - 2 b_1 b_2 \\ + 9 a_3 c_0 \\ - a_1 a_2 c_0 \end{array} & \begin{array}{l} a_2 b_0 b_2 \\ - a_3 b_0 b_1 \\ - b_2^2 \\ + 3 a_1 a_3 c_0 \\ - a_2^2 c_0 \end{array} & 0 \\ 0 & 0 & \begin{array}{l} a_1 b_0 b_1 \\ - b_1^2 \\ + b_0 b_2 \\ - a_2 b_0^2 \\ + 3 a_2 c_0 \\ - a_1^2 c_0 \end{array} & \begin{array}{l} 2 a_1 b_0 b_2 \\ - 2 a_3 b_0^2 \\ - 2 b_1 b_2 \\ + 9 a_3 c_0 \\ - a_1 a_2 c_0 \end{array} & \begin{array}{l} a_2 b_0 b_2 \\ - a_3 b_0 b_1 \\ - b_2^2 \\ + 3 a_1 a_3 c_0 \\ - a_2^2 c_0 \end{array} \end{vmatrix} = 0,$$

$$J_3 = \begin{vmatrix}
1 & a_1 & a_2 & a_3 & 0 \\
0 & 1 & a_1 & a_2 & a_3 \\
\begin{matrix} a_1 b_0 b_2 \\ - a_3 b_0^2 \\ - b_1 b_2 \\ - a_1^2 c_1 \\ + 3 a_2 c_1 \end{matrix} & \begin{matrix} 2 a_2 b_0 b_2 \\ - 2 a_3 b_0 b_1 \\ - 2 b_2^2 \\ + 9 a_3 c_1 \\ - a_1 a_2 c_1 \end{matrix} & \begin{matrix} a_2 b_1 b_2 \\ - a_1 b_2^2 \\ - a_2^2 c_1 \\ + 3 a_1 a_3 c_1 \\ + a_3 b_0 b_2 \\ - a_3 b_1^2 \end{matrix} & 0 & 0 \\
0 & \begin{matrix} a_1 b_0 b_2 \\ - a_3 b_0^2 \\ - b_1 b_2 \\ - a_1^2 c_1 \\ + 3 a_2 c_1 \end{matrix} & \begin{matrix} 2 a_2 b_0 b_2 \\ - 2 a_3 b_0 b_1 \\ - 2 b_2^2 \\ + 9 a_3 c_1 \\ - a_1 a_2 c_1 \end{matrix} & \begin{matrix} a_2 b_1 b_2 \\ - a_1 b_2^2 \\ - a_2^2 c_1 \\ + 3 a_1 a_3 c_1 \\ + a_3 b_0 b_2 \\ - a_3 b_1^2 \end{matrix} & 0 \\
0 & 0 & \begin{matrix} a_1 b_0 b_2 \\ - a_3 b_0^2 \\ - b_1 b_2 \\ - a_1^2 c_1 \\ + 3 a_2 c_1 \end{matrix} & \begin{matrix} 2 a_2 b_0 b_2 \\ - 2 a_3 b_1 b_2 \\ - 2 b_2^2 \\ + 9 a_3 c_1 \\ - a_1 a_2 c_1 \end{matrix} & \begin{matrix} a_2 b_1 b_2 \\ - a_1 b_2^2 \\ - a_2^2 c_1 \\ + 3 a_1 a_3 c_1 \\ + a_3 b_0 b_2 \\ - a_3 b_1^2 \end{matrix}
\end{vmatrix} = 0.$$

CASE 4. $d_{03}^{5} \neq 0$. By a simple real translation of the y-axis this case is reduced back to one of the first three cases.

We consider next the algebraical conditions for degenerate surfaces. Let

$$f_{4m} = a_x^m + b_x^{m-1} + \dots + k_x + l,$$

where

$$a_x^m = \sum_{j=0}^{m} (a_{0\,m-j} x_1^{m-j} + a_{1\,m-j} x_1^{m-j-1} x_2 + \dots + a_{m-j\,m-j} x_2^{m-j}) x_3^j \quad (a_{0m} = 1),$$

$$b_x^{m-1} = \sum_{i=0}^{m-1} (b_{0\,m-i-1} x_1^{m-i-1} + b_{1\,m-i-1} x_1^{m-i-2} x_2 + \dots + b_{m-i-1\,m-i-1} x_2^{m-i-1}) x_3^i,$$

....

Suppose that f_{4m} is degenerate, the component forms being $\varkappa$ linear and $\lambda = \frac{1}{2}(m - \varkappa)$ quadratic forms. Then the n_1 conditions for the decomposition of the form a_x^m into $\varkappa$ linear and λ quadratic ternary forms must be fulfilled, each

$$L_8 = b_{01}(a_{13}^2 a_{23}^2 + 18\, a_{13} a_{23} a_{33} - 4\, a_{23}^3 - 4\, a_{13}^3 a_{33} - 27\, a_{33}^2)^{\frac{1}{2}} + b_{22}(12\, a_{23} a_{01} - 3\, a_{12}^2 - 4\, a_{02}^2 a_{23} + 4\, a_{13} a_{02} a_{12} - 4\, a_{13}^2 a_{01})^{\frac{1}{2}} + 15^{-\frac{1}{2}} b_{12}(-14\, a_{13}^4 a_{01} + 14\, a_{13}^3 a_{02} a_{12} - 14\, a_{13}^2 a_{02}^2 a_{23} + 52\, a_{13}^2 a_{23} a_{01} - 14\, a_{13}^2 a_{12}^2 + 11\, a_{02}^2 a_{23}^2 + 4\, a_{13} a_{02} a_{23} a_{12} + 14\, a_{13}^3 a_{11} - 28\, a_{13}^2 a_{02} a_{22} - 18\, a_{13} a_{02}^2 a_{33} - 44\, a_{23}^2 a_{01} - 4\, a_{23} a_{12}^2 + 12\, a_{02} a_{12} a_{33} + 42\, a_{13} a_{01} a_{33} + 22\, a_{02} a_{23} a_{22} + 42\, a_{13} a_{12} a_{22} + 52\, a_{13} a_{23} a_{11} - 33\, a_{22}^2 - 36\, a_{33} a_{11})^{\frac{1}{2}} + b_{02}(a_{13}^2 a_{22}^2 - 2\, a_{13} a_{02} a_{33} a_{22} + 3\, a_{02}^2 a_{33}^2 - 2\, a_{13}^3 a_{01} a_{33} + 2\, a_{13}^2 a_{02} a_{12} a_{33} - 2\, a_{13} a_{02}^2 a_{23} a_{33} - 2\, a_{13} a_{12}^2 a_{33} + 8\, a_{13} a_{23} a_{01} a_{33} - 2\, a_{13}^2 a_{11} a_{33} + 2\, a_{23} a_{11} a_{33} + 2\, a_{12} a_{22} a_{33} - 14\, a_{01} a_{33}^2 - 4 \Sigma r_1^3 r_2^3 s_3^2)^{\frac{1}{2}} + b_{02}(a_{23}^2 a_{12}^2 + a_{13}^2 a_{22}^2 - a_{02}^2 a_{33}^2 - 2\, a_{13} a_{23} a_{12} a_{22} - 2\, a_{02} a_{23} a_{12} a_{33} - 2\, a_{01} a_{33}^2 + 2\, a_{13} a_{02} a_{33} a_{22} - 2\, a_{13}^3 a_{01} a_{33} + 2\, a_{13}^2 a_{02} a_{12} a_{33} - 2\, a_{13} a_{02}^2 a_{23} a_{33} - 2\, a_{13} a_{12}^2 a_{33} + 8\, a_{13} a_{23} a_{01} a_{33} - 2\, a_{13}^2 a_{11} a_{33} + 2\, a_{23} a_{11} a_{33} + 2\, a_{12} a_{22} a_{33} - 4 \Sigma r_1^3 r_2^3 s_1 s_2)^{\frac{1}{2}} = 0.$$

The determinant L_9 is given by operating upon L_8 the substitution $\begin{pmatrix} a_{j\,3-i} & b_{01} & b_{02} & b_{12} & b_{22} \\ a_{i\,3-j} & b_{12} & b_{02} & b_{01} & b_{00} \end{pmatrix}$.

$$L_{10} = b_{11}(12 a_{13} a_{01} - 3\, a_{12}^2 - 4\, a_{02}^2 a_{23} + 4 a_{13} a_{02} a_{12} - 4 a_{13}^2 a_{01})^{\frac{1}{2}} + 15^{-\frac{1}{2}} b_{01}(-64\, a_{13}^4 a_{01} + 64\, a_{13}^3 a_{02} a_{12} - 64\, a_{13}^2 a_{02}^2 a_{23} + 252\, a_{13}^2 a_{23} a_{01} - 49\, a_{13}^2 a_{12}^2 - 26\, a_{13} a_{02} a_{23} a_{12} + 11\, a_{02}^2 a_{23}^2 + 4\, a_{13}^3 a_{11} - 68\, a_{13}^2 a_{02} a_{22} + 132\, a_{13} a_{02}^2 a_{33} - 44\, a_{23}^2 a_{01} - 4\, a_{23} a_{12}^2 + 12\, a_{02} a_{12} a_{33} - 408\, a_{13} a_{01} a_{33} + 22\, a_{02} a_{23} a_{22} + 102\, a_{13} a_{12} a_{22} - 33\, a_{22}^2 + 82\, a_{13} a_{23} a_{11} - 36\, a_{33} a_{11})^{\frac{1}{2}} + 15^{-\frac{1}{2}} b_{12}(-14\, a_{02}^4 a_{23} + 14\, a_{02}^3 a_{13} a_{12} - 14\, a_{02}^2 a_{13}^2 a_{01} + 52\, a_{02}^2 a_{01} a_{23} - 14\, a_{02}^2 a_{12}^2 + 4\, a_{02} a_{13} a_{01} a_{12} + 11\, a_{13}^2 a_{01}^2 + 14\, a_{02}^3 a_{22} - 28\, a_{02}^2 a_{13} a_{11} - 18\, a_{02} a_{13}^2 a_{00} - 44\, a_{01}^2 a_{23} - 4\, a_{01} a_{12}^2 + 12\, a_{13} a_{12} a_{00} + 42\, a_{02} a_{23} a_{00} + 22\, a_{13} a_{01} a_{11} + 42\, a_{02} a_{12} a_{11} - 33\, a_{11}^2 + 52\, a_{02} a_{01} a_{22} - 36\, a_{00} a_{22})^{\frac{1}{2}} + 15^{-\frac{1}{2}} b_{12}(-64\, a_{02}^4 a_{23} + 64\, a_{02}^3 a_{13} a_{12} - 64\, a_{02}^2 a_{13}^2 a_{01} + 252\, a_{02}^2 a_{01} a_{23} - 49\, a_{02}^2 a_{12}^2 - 26\, a_{02} a_{13} a_{01} a_{12} + 11\, a_{13}^2 a_{01}^2 + 4\, a_{02}^3 a_{22} - 68\, a_{02}^2 a_{13} a_{11} + 132\, a_{02} a_{13}^2 a_{00} - 44\, a_{01}^2 a_{23} - 4\, a_{01} a_{12}^2 + 12\, a_{13} a_{12} a_{00} - 408\, a_{02} a_{23} a_{00} + 22\, a_{13} a_{01} a_{11} + 102\, a_{02} a_{12} a_{11} - 33\, a_{11}^2 + 82\, a_{02} a_{01} a_{22} - 36\, a_{00} a_{22})^{\frac{1}{2}} + 15^{-\frac{1}{2}} b_{01}(-14 a_{13}^4 a_{01} + 14\, a_{13}^3 a_{02} a_{12} - 14\, a_{13}^2 a_{02}^2 a_{23} + 52\, a_{13}^2 a_{23} a_{01} - 14\, a_{13}^2 a_{12}^2 + 4\, a_{13} a_{02} a_{23} a_{12} + 11\, a_{02}^2 a_{23}^2 + 14\, a_{13}^3 a_{11} - 28\, a_{13}^2 a_{02} a_{22} - 18\, a_{13} a_{02}^2 a_{33} - 44\, a_{23}^2 a_{01} - 4\, a_{23} a_{12}^2 + 12\, a_{02} a_{12} a_{33} + 42\, a_{13} a_{01} a_{33} + 22\, a_{02} a_{23} a_{22} + 42\, a_{13} a_{12} a_{22} - 33\, a_{22}^2 + 52\, a_{13} a_{23} a_{11} - 36\, a_{33} a_{11})^{\frac{1}{2}} + 3^{-1} b_{02}(-36 \Sigma r_1^4 s_2^2 s_3^2 - 36 \Sigma r_1^2 r_2^2 s_3^4 - 36 \Sigma r_1^2 r_2^2 s_1^2 s_2^2 + (-4\, a_{13}^2 a_{01} + 4\, a_{13} a_{02} a_{12} - 4\, a_{02}^2 a_{23} - a_{12}^2 + 10\, a_{23} a_{01} - 2 a_{13} a_{11} - 2\, a_{02} a_{22})^2)^{\frac{1}{2}} + \tfrac{2}{3} b_{02}((-a_{13}^2 a_{01} + a_{13} a_{02} a_{12} - a_{02}^2 a_{23} - a_{12}^2 + 4\, a_{23} a_{01} + a_{13} a_{11} + a_{02} a_{22})^2 - 36\,(a_{00} a_{22} a_{23} + a_{00} a_{02} a_{13} a_{33} - 3\, a_{00} a_{12} a_{33} + a_{01} a_{11} a_{33}))^{\frac{1}{2}} = 0.$$

§2. *Degenerate and Penultimate Loci.*

The coefficients of the satellite function μ_{m-2} in

$$f = \Pi\, \zeta_{i r_i} + \mu_{m-2} = 0$$

are rational functions of the coefficients of f, and linear functions of the

$\Sigma \frac{1}{2} \tau_i (\tau_i - 1)$ arbitrary coefficients R of $\zeta_{i\tau_i}$. The set of equations obtained by equating μ_{m-3} identically to zero can sometimes be satisfied in only a finite number of ways, in which event the curve (surface) $f = 0$ has only a finite number of degenerate forms.

For example, the cubic

$$(x - \alpha y)(y - \beta z)(z - \gamma x) + (x - \alpha z)(y - \beta x)(z - \gamma y) = 0$$

reduced to the form (31) is

$$\begin{aligned}[]
&[\gamma(\alpha-1)x + \alpha(\gamma-1)][\gamma(\beta-1)y + \beta(\gamma-1)][\gamma(\beta-1)x + \gamma(\alpha-1)y + R] \\
&\qquad - \left[R - \gamma\frac{(\alpha-1)(\beta-1)}{\gamma-1}\right][\beta\gamma(\alpha-1)(\gamma-1)x + \alpha\gamma(\beta-1)(\gamma-1)y] \\
&\qquad - \alpha\beta(\gamma-1)^2[\gamma(\beta-1)x + \gamma(\alpha-1)y + R] = 0.
\end{aligned}$$

Hence, there are two degenerate forms of the curve corresponding to the two cases

(a) $R\left(= 2(1-\alpha\beta\gamma) - \frac{\alpha(\beta-1)(\gamma-1)}{\alpha-1} - \frac{\beta(\alpha-1)(\gamma-1)}{\beta-1}\right) = \frac{\gamma(\alpha-1)(\beta-1)}{\gamma-1}$,

(b) $R = 0,\ \alpha(\beta-1)^2 = \beta(\alpha-1)^2 = \frac{\alpha\beta}{\gamma}(\gamma-1)^2$.

They are respectively*

$$\left(\frac{x}{\alpha-1} + \frac{y}{\beta-1} + \frac{1}{\gamma-1}\right)\left(\frac{\beta}{\beta-1}x + \frac{\alpha}{\alpha-1}y + \frac{\gamma}{\gamma-1}xy\right) = 0,$$

$$\left(\frac{\gamma}{\gamma-1}x + \frac{\alpha}{\alpha-1}\right)\left(\frac{\gamma}{\gamma-1}y + \frac{\beta}{\beta-1}\right)\left(\frac{\beta}{\beta-1}x + \frac{\alpha}{\alpha-1}y\right) = 0.$$

When the general cubic curve $f_{3m} = 0$ is treated in this manner, we find six degenerate forms, three consisting of triads of lines, and three of a line and a conic.

It is to be observed that each degenerate (ultimate) form of a curve or suface has corresponding to it a large number of penultimate forms. That is, a locus can pass over into any degenerate form in a number of different ways; and it is possible to enumerate all the penultimate curves which correspond to any determined degenerate form of a curve. Thus the penultimate quartic curve

$$\zeta_1 \zeta_2 \zeta_3 \zeta_4 + \varepsilon^2 \mu_2 = 0 \quad (\varepsilon \doteq 0)$$

crosses its asymptotes ζ_i at those points in which the latter are cut by the satellite conic $\mu_2 = 0$, and in no other points. Let this conic be chosen such that it encloses all six vertices of the quadrilateral formed by the asymptotes. Then

$$\zeta_1 = -\varepsilon^2 \mu_2 / \zeta_2 \zeta_3 \zeta_4 .$$

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* Cf. Taylor, *loc. cit.*

Hence, for points of the curve approximating to the segment $\overline{ab}$, ζ_2, ζ_3, ζ_4 are negative, μ_2 positive; so that the curve is on the side of $\overline{ab}$ opposite from the origin. In this manner the curve may be traced completely. Its shape in the vicinity of the vertex c should be noted particularly.

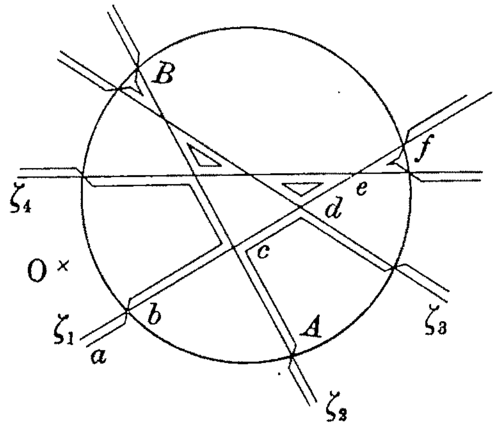

FIG. 1.

Now let the satellite conic vary in position, its perimeter approaching c and crossing it, while the relative positions of the conic, the origin and the other five vertices remain unaltered. When $\mu_2 = 0$ crosses c, the latter will be a double point of the penultimate curve; and when c becomes exterior to the curve $\mu_2 = 0$, the shape of the quartic near c is as shown in Fig. 2. Thus, by variation of the

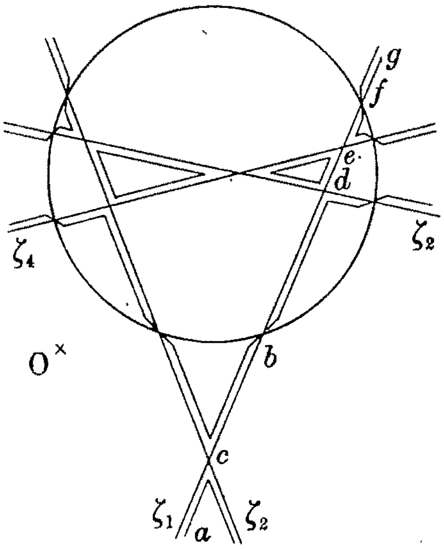

FIG. 2.

position of the satellite conic three distinct cases are produced at each vertex, making 3^6 ways of drawing curves penultimate to four straight lines. Not all of these types are distinct as types. Also Plücker's laws must be satisfied. It is found, for example, that the number of distinct types of penultimate quartics with four rectilinear asymptotes, and having no point singularities at infinity, is 343.

Consider also quartics having two asymptotic ellipses intersecting in four real points

$$\zeta_{12}\zeta_{22} - \varepsilon^2\mu_2 = 0 \quad (\varepsilon \doteq 0).$$

If the conic $\mu_2 = 0$ encloses all of these points, the quartic is the four-ovaled type (Fig. 3). If the satellite encloses none of the points, the penultimate curve is the type having one nested oval. One of these types may be made to pass into the other by continuous variation, by small variations of the coefficients of the satellite $\mu_2 = 0$.

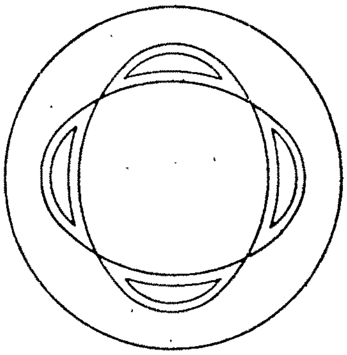

FIG. 3.

Analogously, along a line of intersection of a pair of asymptotic planes of any penultimate surface there are three possible cases of the relative positions of surface and planes. Consider the penultimate cubic surfaces having three asymptotic planes L, M, N intersecting in the point C and lines $\overline{CA}$, $\overline{CB}$, $\overline{CD}$. Let the satellite plane intersect $\overline{CA}$, $\overline{CB}$, $\overline{CD}$ respectively in the points A, B, D Let the origin O be on the side of the satellite plane $\overline{ABD}$ opposite from C and inside the trihedral angle $\overline{ABD} - C$. Then along the line $\overline{CA}$ from infinity up to A the cubic surface is in the dihedral angle which contains the origin, and its vertical. But from A to C the surface lies in the other pair of vertical dihedral angles. Similarly the surface may be traced along the other asymptotic planes. It hangs together at the points A, B, D and at no other points. By shifting the satellite plane into all typical positions with reference to the origin and the asymptotic planes, we get all the possible types of cubic surface penultimate to three planes.

§ 3. *Penultimate Class Loci.*

The case where, in f_{3m}, a_x^m has equal roots, excluded in most of the previous discussion, is best treated from the standpoint of the line equation. This subject we may introduce by the following quotation from Cayley:* "The system of tangents from an arbitrary point to the penultimate curve reduces (when the curve approaches degeneracy) to 1) the tangents to the several component curves; 2) the lines through the singular points of these same curves respectively; 3) the lines through the points of intersection of each two of the component curves; these points, each reckoned the proper number of times, are called 'fixed summits'; 4) the lines from the arbitrary point to certain determinate points called 'free summits' on the several component curves . . . . Thus a degenerate form of the n-ic curve . . . . may be regarded as consisting of the component curves, each its proper number of times, and the foregoing points called summits, and is consequently only inadequately represented by the ultimate form of the equation ($\zeta_{1\tau_1}\zeta_{2\tau_2}\cdots\cdots\zeta_{\nu\tau_\nu}=0$)."

The line equation of $f_{3m}=0$, say $\phi_{3n}=0$, can, by the methods of Part I, be thrown into the form

$$\phi_{3n}=\eta_{1\rho_1}\eta_{2\rho_2}\cdots\cdots\eta_{\mu\rho_\mu}+\psi_{n-3}=0.$$

Keeping the restriction $\tau_i=1$ or 2, we have $\rho_j=1$ or 2. Then the $\eta_{j\rho_j}=0$ ($\rho_j=2$) are the line equations of the asymptotic conics of the penultimate $f_{3m}=0$. But the linear forms η can represent only *points on* the linear constituents of the penultimate $f_{3m}=0$. These points are in fact the before-mentioned "summits" of the curve. Thus the cubic

$$\begin{aligned}f_{33}=x_1^3+x_2^3+p\,x_1+v\,x_2&=(x_1+x_2)(x_1^2-x_1x_2+x_2^2+R)\\&\quad+(p-R)\,x_1+(v-R)\,x_2=\zeta_{11}\zeta_{22}+\mu_1=0\end{aligned}$$

degenerates when $R=p=v$ into line and ellipse $\zeta_{11}\zeta_{22}=0$. The class equation of $\zeta_{11}\zeta_{22}=0$ is

$$\begin{aligned}\phi_{33}=\eta_{11}^2\eta_{21}^2\eta_{32}=\left(u_1-u_2+\sqrt{\frac{3}{-v}}\,u_3\right)^2\left(u_1-u_2-\sqrt{\frac{3}{-v}}\,u_3\right)^2&\\(4\,v\,u_1^2+4\,v\,u_2^2+4\,v\,u_1u_2+3\,u_3^2)=0,&\end{aligned}$$

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* Cayley, *loc. cit.*

and $\eta_{33}=0$ is the class equation of $\zeta_{22}=0$, whereas $\eta_{11}^2=0$, $\eta_{21}^2=0$ are the points of intersection of $\zeta_{11}=0$ and $\zeta_{22}=0$, the fixed summits, each counting double. Conditions similar to the above hold when $\tau_i > 2$.

It is well-known that the class equation of a curve $f=0$ can be reduced to the form*

$$\rho_1\rho_2\cdots\cdot\rho_n+\omega\omega'\psi'_{n-2}=0,$$

where the ρ_i are the foci of the curve and ω, ω' are the circular points at infinity. When the non-linear factors of the degenerate ϕ_{3n} are each put in this form, we have

$$\phi_{3n}=\eta_{11}\eta_{21}\cdots\cdot\eta_{\mu 1}\rho_1\rho_2\cdots\cdot\rho_1'\rho_2'\cdots\cdot+\omega\omega'\psi''_{n-2}=0.$$

Hence, *when by small variations of its coefficients the class locus* $\phi_{3n}=0$ *degenerates, it degenerates into those curves which its asymptotes approach as limiting curves, together with those points which its foci approach as limiting points, during the variation, the foci of the non-linear constituents being excepted.*

Whenever a penultimate curve $f_{3m}=0$ possesses a proper class equation $\phi_{3n}=0$, the latter's linear constituents represent only *fixed* summits (and double points). The function ϕ_{3n} vanishes identically only in case the penultimate f_{3m} has a multiple asymptote. Therefore *free summits can exist only upon penultimate curves having multiple asymptotes.* Since in this case the class equation is evanescent and the degree equation gives no evidence of the positions of the summits, the analytical determination of the free summits depends upon special methods. Cayley and Zeuthen have published tentative methods of finding the number of these summits. By using symbolical methods, the details of which are too long for the scope of this paper, I have developed a method of finding the multiplicities of the fixed summits. If n_i $(i=1, 2, \ldots.)$ represents the number of fixed summits of species 1, 2, ... and m_i the multiplicity of n_i, also if q is the number of free summits and the classes of the non-linear (and non-singular, say) constituents of ϕ_{3n} are respectively r_1, r_2, ..., we have of course a relation

$$q=n-\Sigma m_i n_i-\Sigma r_j.$$

Zeuthen has in certain special problems determined the equation of a curve which intersects the multiple asymptotes in the free summits, thus determining geometrically the positions of the latter. It should be noted that, in general, *the intersections of the satellite* $\mu_{m-2}=0$ *with the multiple asymptotes of the penultimate curve* $f_{3m}=0$ *are all free summits of the latter curve.*

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* C. A. Scott, Modern Analytic Geometry, § 270.

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The Reduction of Families of Bilinear Forms.

BY H. E. HAWKES.

Introduction

The problem of determining the necessary and sufficient conditions that two families of non-singular bilinear forms are equivalent was first solved by Weierstrass in 1868. Since that time the subject has received considerable attention from various investigators, notably Frobenius, Kronecker, Weyr and Bromwich, but until recently a connected and clear account of this theory has been lacking. The work of Bromwich, and in greater measure Bôcher's Introduction to Higher Algebra, have served to supply the deficiency, so that the theory cannot now be called inaccessible.

The present paper gives a direct and practically self-contained derivation of the necessary and sufficient condition that two non-singular families of bilinear forms are equivalent by a method that is not only simpler of application than the classical methods, but well adapted for purposes of exposition. The simplicity of the demonstration rests in referring the general question of equivalence of families back to that of the similarity of numerical matrices. The question of similarity is settled by the derivation of a normal form in which a complete system of invariants of a class of matrices is displayed.

§ 1.

Two families of bilinear forms in $2n$ variables,

$$\lambda' A + \lambda'' B = \Sigma(\lambda' a_{ij} + \lambda'' b_{ij}) x_i y_j, \qquad (1)$$

$$(i, j = 1, 2, \ldots, n)$$

$$\lambda' A' + \lambda'' B' = \Sigma(\lambda' a'_{ij} + \lambda'' b'_{ij}) x'_i y'_j, \qquad (2)$$

are equivalent when and only when linear transformations,

$$x_i = \sum_l h_{il} x'_l \qquad |h_{il}| \ddagger 0,$$

$$y_j = \sum_k k_{jk} y'_k \qquad |k_{jk}| \ddagger 0,$$

exist that will transform (1) into (2). In these families, the parameters λ' and λ'', and the coefficients, are assumed to be real or complex numbers. This transformation may be expressed as follows in the notation of matrices:

$$\mathbf{h}(\lambda'\mathbf{a} + \lambda''\mathbf{b})\mathbf{k} = \lambda'\mathbf{a}' + \lambda''\mathbf{b}', \tag{3}$$

where $\mathbf{h}$, $\mathbf{a}$, etc. symbolize the matrices whose determinants are $|h_{ii}|$, $|a_{ij}|$, etc.

The necessary and sufficient conditions for the validity of (3) are that

$$\mathbf{hak} = \mathbf{a}'; \quad \mathbf{hbk} = \mathbf{b}'. \tag{4}$$

The sufficiency of these conditions follows immediately from the application of the distributive law in operations with matrices; their necessity, from the fact that (3) is valid for all values of λ' and λ''.

We proceed to find a method of determining the conditions under which $\mathbf{h}$ and $\mathbf{k}$ exist; that is, simple means of determining whether (1) and (2) are equivalent. Let us first consider the exceptional case when $|\mathbf{a}| \equiv |\mathbf{b}| \equiv 0$; while $|\lambda'\mathbf{a} + \lambda''\mathbf{b}| \equiv |\mathbf{c}| \not\equiv 0$. For a certain pair of values of λ' and λ'' for which the determinant $|\mathbf{c}|$ does not vanish, let $\lambda'\mathbf{a}' + \lambda''\mathbf{b}' \equiv \mathbf{c}'$. Then if $\mathbf{c}$ and $\mathbf{c}'$ are equivalent,

$$\mathbf{hck} = \mathbf{c}'; \quad \text{and} \quad \mathbf{hak} = \mathbf{a}'.$$

Consequently

$$\mathbf{k} = \mathbf{c}^{-1}\mathbf{h}^{-1}\mathbf{c}', \quad \text{and} \quad \mathbf{a}' = \mathbf{hac}^{-1}\mathbf{h}^{-1}\mathbf{c}',$$

or

$$\mathbf{a}'\mathbf{c}'^{-1} = \mathbf{h}(\mathbf{ac}^{-1})\mathbf{h}^{-1}; \tag{5}$$

thus if $\mathbf{c}$ and $\mathbf{c}'$ are equivalent, the matrix $\mathbf{a}'\mathbf{c}'^{-1}$ is similar* to the matrix $\mathbf{ac}^{-1}$. The converse is also true. For from (5), letting $\mathbf{k} = \mathbf{c}^{-1}\mathbf{h}^{-1}\mathbf{c}'$, we obtain immediately

$$\mathbf{a}' = \mathbf{hak}; \quad \mathbf{c}' = \mathbf{hck}.$$

This criterion also holds in the unexceptional case where at least one of the determinants $|\mathbf{a}|$ and $|\mathbf{b}|$, say $|\mathbf{b}|$, does not vanish. In this case the condition (5) is replaced by the condition

$$\mathbf{a}'\mathbf{b}'^{-1} = \mathbf{h}(\mathbf{ab}^{-1})\mathbf{h}^{-1}. \tag{6}$$

The case where $|\lambda'\mathbf{a} + \lambda''\mathbf{b}| \equiv 0$, which Kronecker has completely discussed, will not be considered in this paper.

The problem of determining the equivalence of families of bilinear forms (1) and (2) is, then, reduced to the problem of determining the similarity of the matrices $\mathbf{a}'\mathbf{c}'^{-1}$ and $\mathbf{ac}^{-1}$. It should be observed that the determinant of these matrices may be zero.

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*Frobenius defines the matrices $\mathbf{a}$ and $\mathbf{b}$ to be similar (ähnlich) when a matrix $\mathbf{l}$ exists such that $\mathbf{a} = \mathbf{lbl}^{-1}$. See *Crelle*, 84, p. 21.

§ 2.

We may, if we choose, think of all the matrices that are similar to each other as forming a class of matrices. If we can determine a complete system of invariants for a class of matrices, we shall be in a position to settle the question of the equivalence of bilinear forms that was stated in the last section. As a matter of fact, we shall prove that the following constitute such a system for a class of matrices under the transformations of similarity:

I. The rank of any matrix, **a**, of the class.
II. The characteristic equation of **a**.
III. The degrees of the elementary divisors of the matrix $\lambda - \mathbf{a}$.

The proof of the first two parts of this statement is very simple and familiar, but will be given for the sake of completeness.

Invariant I. Given a matrix **a** of order n and rank k. Let $\mathbf{b} = \mathbf{hah}^{-1}$. Then since the rank of the product of determinants cannot exceed the lowest rank of the factors, the rank of **b** cannot be greater than that of **a**. Similarly, since $\mathbf{a} = \mathbf{h}^{-1}\mathbf{bh}$, and the rank of **a** cannot exceed that of **b**. Hence the rank is invariant.

Invariant II. By the characteristic equation of a matrix

$$\mathbf{a} = \left\|\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{32} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{matrix}\right\|$$

is meant the equation of the nth degree in λ that one obtains by developing the determinant in the equation

$$\phi(\lambda) = \left|\begin{matrix} \lambda - a_{11} & a_{12} & a_{13} \cdots\cdots\cdots & a_{1n} \\ a_{21} & \lambda - a_{22} & a_{23} \cdots\cdots\cdots & a_{2n} \\ a_{31} & a_{32} & \lambda - a_{33} \cdots\cdots\cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} \cdots & \lambda - a_{nn} \end{matrix}\right| = 0.$$

Suppose this equation has the form

$$\phi(\lambda) = \lambda^n + \alpha_1\lambda^{n-1} + \alpha_2\lambda^{n-2} + \cdots + \alpha_n = 0.$$

The theorem is well-known that the matrix **a** satisfies this equation identically, that is, $\phi(\mathbf{a}) \equiv 0$. From this it follows that the characteristic equation is an invariant of the class of similar matrices. For it is only necessary to show that the equation $\phi(\lambda) = 0$ is satisfied identically when $\mathbf{hah}^{-1}$ is substituted for λ. This fact appears immediately, for

$$\begin{aligned}\phi(\mathbf{hah}^{-1}) &= (\mathbf{hah}^{-1})^n + \alpha_1(\mathbf{hah}^{-1})^{n-1} + \ldots . + \alpha_n \\ &= \mathbf{ha}^n\mathbf{h}^{-1} + \alpha_1\mathbf{ha}^{n-1}\mathbf{h}^{-1} + \ldots . + \alpha_n = \mathbf{h}\phi(\mathbf{a})\mathbf{h}^{-1} \equiv 0.^*\end{aligned}$$

Invariant III. We now proceed to define the elementary divisors of the λ-matrix $\lambda - \mathbf{a}$. As we have seen, the determinant of this matrix is a rational integral equation in λ, and is consequently factorable into n linear factors, $(\lambda - \lambda_1)^{l_1}(\lambda - \lambda_2)^{l_2}(\lambda - \lambda_3)^{l_3} \ldots . (\lambda - \lambda_k)^{l_k}$, where $l_1 + l_2 + l_3 + \ldots . + l_k = n$. Suppose that $(\lambda - \lambda_1)^{l'_1}$ is the highest power of $\lambda - \lambda_1$ that is contained in all the first minors of the determinant $|\lambda - \mathbf{a}|$, that l''_1 is the highest exponent of this same factor that is found in all the second minors, and that, continuing in the same way, $l_1^{\tau_1}$ is the highest exponent of $\lambda - \lambda_1$ that is found in all the τth minors of the determinant. Similarly, we may obtain sets of exponents bearing the same relation to the other factors of the original determinant, such as

$$l'_2,\ l''_2,\ l'''_2, \ldots ,\ l_2^{\tau_2};\ \ldots ;\ l'_k,\ l''_k,\ l'_k,\ \ldots ,\ l_k^{\tau_k}.$$

Now the first differences of these exponents,

$$l_1 - l'_1 = e_1,\ l'_1 - l''_1 = e_2,\ l''_1 - l'''_1 = e_3,\ \ldots ,\ l_1^{\tau-1} - l_1^{\tau} = e_\tau,$$

are called the exponents of the elementary divisors of the matrix. The expressions $(\lambda - \lambda_1)^{e_1}$, $(\lambda - \lambda_1)^{e_2}$, $\ldots$, $(\lambda - \lambda_1)^{e_\tau}$ are called the elementary divisors of the matrix $\lambda - \mathbf{a}$, corresponding to the factor $\lambda - \lambda_1$. We can obtain the complete set of elementary divisors of the matrix by considering the other factors of the characteristic equation.

We shall, however, be interested chiefly in the exponents e_1, e_2, e_3, ….. It appears that if the roots of the characteristic equation and these e's, together with their distribution over the various factors, are known, we then know all the elementary divisors of the matrix. If, then, we prove that the elementary divisors are invariants of a class of λ-matrices, the only actual addition to our two invariants I and II are these e's and their distribution on the various factors of the characteristic equation.

The proof that the elementary divisors of a matrix are invariants of the matrix follows from the theorem that the τth minors of the product of two

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[*] Of course this same demonstration would show that any equation satisfied by $\mathbf{a}$ is an invariant of the class of matrices.

determinants are linear in each of the rth minors of the factors. Hence the elementary divisors of the matrix $\lambda - \mathbf{a}$, and of any matrix equivalent to it, as $\mathbf{s}(\lambda - \mathbf{a})\mathbf{t} = \lambda - \mathbf{b}$, are the same.

Now if $\mathbf{a}$ and $\mathbf{b}$ are similar, $\lambda - \mathbf{a}$ and $\lambda - \mathbf{b}$ are equivalent, and conversely.

For suppose that $\mathbf{a}$ and $\mathbf{b}$ are similar. Then the matrices $\lambda - \mathbf{a}$ and $\lambda - \mathbf{b}$ are equivalent. For if $\mathbf{b} = \mathbf{hah}^{-1}$, then evidently $\lambda - \mathbf{b} = \mathbf{h}(\lambda - \mathbf{a})\mathbf{h}^{-1}$, since $\mathbf{h}\lambda\mathbf{h}^{-1} = \lambda$. Conversely, if $\lambda - \mathbf{a}$ and $\lambda - \mathbf{b}$ are equivalent, then $\mathbf{a}$ and $\mathbf{b}$ are similar. For, suppose that $\mathbf{s}(\lambda - \mathbf{a})\mathbf{t} = \lambda - \mathbf{b}$. Since the distributive law holds in the operations with matrices, we have $\mathbf{s}\lambda\mathbf{t} - \mathbf{sat} = \lambda - \mathbf{b}$, or since this equation is identically true, $\mathbf{s}\lambda\mathbf{t} = \lambda$, that is, $\mathbf{st} = 1$, or $\mathbf{t} = \mathbf{s}^{-1}$. Hence $\mathbf{b} = \mathbf{s}^{-1}\mathbf{as}$.

Consequently in determining whether the matrices mentioned at the end of § 1 are similar or not, it is only necessary to discuss the equivalence of the corresponding λ-matrices.

§ 3.

This completes the demonstration that I, II and III are invariants of a class of matrices. It is a more serious matter to show that they form a complete system of invariants. To do this, we shall show how to reduce the matrix $\mathbf{a}$ into a similar normal form which is completely characterized by the system of invariants I, II and III.

Let the roots of $|\lambda - \mathbf{a}| = 0$ be $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$, of which suppose that the first r are distinct. Let the solutions of the r sets of n linear equations in n variables

$$\sum_k a_{lk} x_k = \lambda_h x_l \quad (l = 1, 2, \ldots, n;\ h = 1, 2, \ldots, r)$$

be $\xi_1^{(h)}\ \xi_2^{(h)}\ \xi_3^{(h)} \ldots \xi_n^{(h)}$. Build, now, a matrix of the nth order of which the first r columns are these ξ's, and of which the other elements are all zero excepting those in the principal diagonal, which is made up of 1's in the places not filled with ξ's. Call this matrix $\mathbf{t}$. Build also $\mathbf{t}^{-1}$, and carry out the multiplication $\mathbf{t}^{-1}\mathbf{at}$. The result of this process by which we pass from $\mathbf{a}$ to a similar matrix changes the form of the matrix materially. In fact, in the new form we have all the elements of the first r columns zero except the elements of the principal diagonal, which are $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_r$. In the transformed matrix consider the $(n - r)$-rowed determinant that is common to the last $n - r$ rows and columns. Proceed with this exactly as was done with the original matrix, only in the matrix that corresponds to $\mathbf{t}$ let the elements in the first r columns all be zero except those in the principal diagonal that fall in those columns, which we place equal to 1.

Continue this process till there are no non-zero elements below the principal diagonal, and the principal diagonal consists of the roots of the characteristic equation.

Numerical Example. Let us reduce to normal form the matrix

$$\mathsf{a}=\begin{Vmatrix} 2 & -2 & 1 & 1 \\ 6 & -3 & 0 & 4 \\ 16 & -3 & -4 & 8 \\ 23 & 4 & -13 & 10 \end{Vmatrix}$$

of which the characteristic equation is

$$\lambda^4-5\lambda^3+9\lambda^2-7\lambda+2=0.$$

By taking the derivative of this equation it appears that 1 is a triple root, and that 2 is the other root.

We will first simplify the matrix by bringing 2 to the leading place, and 1 to the next place in the principal diagonal, making all the other elements of the first two columns zero. To this end we get as solutions of the equations

$$\begin{aligned} 2x-2y+z+w&=2x,\\ 6x-3y+4w&=2y,\\ 16x-3y-4z+8w&=2z,\\ 23x+4y-13z+10w&=2w \end{aligned}$$

the values $x=1$, $y=2$, $z=3$, $w=1$. We must also solve the equations that we obtain from these by replacing the right-hand members, viz., $2x$, $2y$, $2z$, $2w$, by x, y, z, w. The solution of these equations is $x=0$, $y=-1$, $z=-1$, $w=-1$. Now compute the product $\mathsf{t}^{-1}\mathsf{a}\mathsf{t}$, where

$$\mathsf{t}=\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{Vmatrix},\quad \mathsf{t}^{-1}=\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{Vmatrix},$$

and we obtain

$$\mathsf{a}=\begin{Vmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & -5 & 3 \\ 0 & 0 & -12 & 7 \end{Vmatrix}.$$

Now solving the equations

$$\begin{aligned} -\ 5x + 3y &= x, \\ -12x + 7y &= y, \end{aligned}$$

we find that $x = 1$, $y = 2$. Now compute $\mathbf{t}_1^{-1}\mathbf{a}\mathbf{t}_1$, where

$$\mathbf{t}_1 = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{Vmatrix},$$

and we obtain

$$\mathbf{a} = \begin{Vmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

From the results that we shall obtain in the next paragraphs it appears that we may replace the last two elements in the first row, and the last element in the second row, by 0 without further transformation. The -2 and the 3 may be replaced by 1, and we have the result

$$\mathbf{a} = \begin{Vmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

Now if a non-zero element, say a_{ik}, occurs in the intersection of the ith row, and the kth column, and if the roots λ_i and λ_k are distinct, this element may be replaced by zero without otherwise disturbing the matrix. For build the matrices $\mathbf{h}$ and $\mathbf{h}^{-1}$ each of which have all the elements in their principal diagonal equal to 1, and which have all the other elements equal to zero except that in the ith row and the kth column, where the values h and $-h$ respectively occur, where $h = \dfrac{a_{ik}}{\lambda_i - \lambda_k}$. The transformation $\mathbf{h}^{-1}\mathbf{a}\mathbf{h}$ does not change the elements of any row above or below the ith, and only the element a_{ik} in that row, which it replaces by 0.

Consequently, since we can always find a transformation that will replace such elements a_{ik}, $(\lambda_i \neq \lambda_k)$ by zero without disturbing the rest of the matrix, there is no necessity of going through the transformation at all, but the elements may be replaced by zero as soon as the previous transformations are completed.

We may now, by an interchange of rows and columns, which in the present form of matrix is equivalent to passing to a similar matrix, bring into juxtaposition along the principal diagonal all the roots that are equal to each other. We have then a matrix of the form

$$\mathbf{a} = \left\| \begin{matrix} \mathbf{a}_1 & 0 & 0 \dots\dots 0 \\ 0 & \mathbf{a}_2 & 0 \dots\dots 0 \\ 0 & 0 & \mathbf{a}_3 \dots\dots 0 \\ \dots & \dots & \dots\dots\dots \\ 0 & 0 & 0 \dots\dots \mathbf{a}_k \end{matrix} \right\|, \tag{7}$$

where $\mathbf{a}_j$ represents a submatrix all of whose roots are equal, and in which all the elements below the principal diagonal are zero.

§ 4.

We may then for the further reduction of the matrix confine ourselves to the consideration of a matrix all of whose roots are equal, and which is in the form

$$\mathbf{a}_1 = \left\| \begin{matrix} \lambda & a_{12} & a_{13} & \dots\dots & a_{1m} \\ 0 & \lambda & a_{23} & \dots\dots & a_{2m} \\ 0 & 0 & \lambda & \dots\dots & a_{3m} \\ \dots & \dots & \dots & \dots\dots & \dots \\ 0 & 0 & 0 & \dots\dots & \lambda \end{matrix} \right\|,$$

where λ is taken to represent any one of the distinct roots $\lambda_1, \lambda_2, \dots, \lambda_k$. We now seek a method of further reducing matrices of this type. To effect this reduction we may use transformations of three types.

First, we may effect an interchange of the subscripts i and j; that is, we may carry out the substitution (ij) on the subscripts by transforming $\mathbf{a}_1$ by the matrix $\mathbf{h}$ in which $h_{ij} = h_{ji} = 1 = h_{kk}$, $(k \neq i, j)$, and all the other elements are zero. Since the transformation is known always to exist, of course the indices may be interchanged without actually going through the transformation.

Second, if $a_{ij} \neq 0$, and at the same time $a_{j-1j} \neq 0$, we may transform $\mathbf{a}_1$ by $\mathbf{h}$, where $h_{kk} = 1$ and all the other elements are zero excepting h_{ij-1}, which has the value a_{ij}/a_{j-1j}. This transformation has the effect of replacing a_{ij} by zero, but not affecting the remainder of the matrix excepting certain elements in the first $i - 1$ rows.

Third, if $a_{ij} \neq 0$, and at the same time $a_{j-1j} = 0$, but $a_{ii+1} \neq 0$, we may transform the matrix $\mathbf{a}_1$ by a transformation of the first type so that a_{ij} falls in the place that was occupied by a_{ii+2}, and is, as a consequence, directly over a zero element. Now transform the matrix by a transformation $\mathbf{h}$ such that $h_{kk} = 1$ and all the other elements are zero excepting h_{i+2i+1}, which has the value $-a_{ii+1}/a_{ii}$. The effect of this transformation is to replace a_{ii+1} by zero, and to leave all the other elements in the matrix unchanged with the exception of certain elements that occur in the first $i - 1$ rows.

By the use of these transformations we are able to start with the last row, and gradually replace all the elements by zero, excepting certain elements that we can always put in the place of the elements a_{ii+1}, and which constitute what we will call the over-principal diagonal. By a simple transformation these elements may all be replaced by unity, so that our matrix $\mathbf{a}_1$ assumes a form in which every element is zero excepting the elements of the principal diagonal, which are all equal to λ, and the elements of the over-principal diagonal, which are all either 1 or 0. This is the normal form for which we were seeking. It appears that any matrix can be brought to this form by properly chosen transformations, and consequently that every matrix is similar to some matrix which is in normal form. It is observed that the matrices that we have considered need not necessarily have the rank n, but the characteristic equation may have zero roots without affecting the method in the least.

The transformation employed in obtaining the form (7) assumed that the characteristic equation had been solved. If this equation is irreducible in the domain of rational numbers, this process is an irrational one. Since the objection to the irrational process is partly a practical one, it is worth while to observe how serious these irrational operations are, practically. In the first place, if there are no multiple roots in the characteristic equation, the fact is determined rationally, and in such a case no reduction is ever necessary, as each of the subdeterminants $\mathbf{a}_i$ in (7) reduces to a single element, and the equivalence hinges on the identity of the characteristic equations themselves, which is surely determined by a rational process. For the case $n = 3$ there can be no irrational multiple root. In the case $n = 4$, the only possibility of having an irrational multiple root is when the characteristic equation is a perfect square; that is, when there are two pairs of conjugate quadratic surd or complex roots. No irrationalities of the third order will be encountered in the reduction of any matrix of lower order than the ninth, and then only when the characteristic equation is a perfect cube; and a matrix of order twenty-five would be the first that would involve non-algebraic irrationalities.

The submatrices in the previous numerical example that correspond to equal roots of the characteristic equation are so simple that it is worth while to give another numerical example of a rather more complicated character. In considering the following matrix of order six, with six equal roots, it is to be noted that the value of λ is assumed to be 1.

$$\mathbf{a}_1 = \begin{Vmatrix} 1 & 1 & 1 & -3 & 5 & -7 \\ 0 & 1 & 1 & -1 & 2 & -5 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$$

Transforming by $\mathbf{h}$ where all the elements of $\mathbf{h}$ are zero except those in the principal diagonal, and the element h_{65}, all of which have the value 1, we obtain

$$\mathbf{a}_1 = \begin{Vmatrix} 1 & 1 & 1 & -3 & -2 & -7 \\ 0 & 1 & 1 & -1 & -3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$$

This is a transformation of the third type.

Interchanging, now, the elements that should be moved when the subscripts 5 and 6 are interchanged, we have

$$\mathbf{a}_1 = \begin{Vmatrix} 1 & 1 & 1 & -3 & -7 & -2 \\ 0 & 1 & 1 & -1 & -5 & -3 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$$

Transforming this matrix by $\mathbf{h}$ where all the elements of $\mathbf{h}$ are zero excepting those in the principal diagonal, and the element h_{34}, all of which have the value 1, we obtain

$$\mathbf{a}_1 = \begin{Vmatrix} 1 & 1 & 1 & -2 & -7 & -2 \\ 0 & 1 & 1 & 0 & -5 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$$

This and the following transformation are of the second type.

Transforming this matrix by $\mathbf{h}$, where all the elements of $\mathbf{h}$ are zero excepting those in the principal diagonal, which have the value 1, and h_{24} which has the value 5, we obtain

$$\mathfrak{a}_1 = \left\Vert \begin{array}{cccccc} 1 & 1 & 1 & 3 & -7 & -2 \\ 0 & 1 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right\Vert .$$

Interchanging the elements corresponding to an interchange of the subscripts 4 and 6, followed by the interchange of 5 and 6, we obtain

$$\mathfrak{a}_1 = \left\Vert \begin{array}{cccccc} 1 & 1 & 1 & -2 & 3 & -7 \\ 0 & 1 & 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right\Vert .$$

Transforming this matrix by $\mathbf{h}$, where the principal diagonal consists of 1's and $h_{43} = \frac{1}{3}$, we get

$$\mathfrak{a}_1 = \left\Vert \begin{array}{cccccc} 1 & 1 & \frac{1}{3} & -2 & -3 & -7 \\ 0 & 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right\Vert .$$

This is a transformation of the third type.

From the nature of the transformations employed it is clear that we may get rid of elements without disturbing the rest of the matrix, so that the normal form of the matrix is finally

$$\mathfrak{a}_1 = \left\Vert \begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right\Vert .$$

It is easy to see that the necessary and sufficient condition that two matrices are similar is that they are reducible to the same normal form. Furthermore, it appears that if we know the rank of a matrix and the roots of its characteristic equation, that is if we have the first two of the invariants mentioned in § 2, we can fix exactly every feature of the normal form of that matrix excepting the arrangement of the 1's along the over-principal diagonal. If, then, we can show that the way these group themselves is determined by the values of the exponents of the elementary divisors, we shall be able to pronounce the set of three invariants a complete system for the class of matrices.

Let the matrix $\mathbf{a}_1$ have the following form:

$$\begin{array}{cccccccccccc}
\lambda & 1 & & & & & & & & & & \\
& \lambda & 1 & \big\} e_1 & & & & & & & & \\
& & \cdot\,\cdot & & & & & & & & & \\
& & & \lambda & 0 & & & & & & & \\
& & & & \lambda & 1 & & & & & & \\
& & & & & \lambda & 1 & \big\} e_2 & & & & \\
& & & & & & \cdot\,\cdot & & & & & \\
& & & & & & & \lambda & 0 & & & \\
& & & & & & & & \lambda & 1 & & \\
& & & & & & & & & \lambda & 1 & \big\} e_k \\
& & & & & & & & & & \cdot\,\cdot & \\
& & & & & & & & & & & \lambda
\end{array}$$

where zeros fill the places left blank, and where the e's represent the number of consecutive 1's in the various groups of 1's in the over-principal diagonal. Suppose that the order of subscripts is so chosen that $e_1 > e_2 > \ldots . > e_k$. Then we can show that these e's are precisely the exponents of the elementary divisors that constitute one of the three invariants mentioned.

Since the matrix $\lambda - \mathbf{a}_1$ only differs from the matrix $\mathbf{a}_1$ in having the elements λ replaced by the elements $\lambda - \lambda_1$ all along the principal diagonal, we may observe the relation between the elementary divisors and the way the 1's are arranged as well from the matrix $\mathbf{a}_1$ that we have just written as from the matrix $\lambda - \mathbf{a}_1$. We wish to observe the highest common factor of the subdeterminants of the various orders of $\mathbf{a}_1$. Consider the highest power of $\lambda - \lambda_1$ that occurs in all the non-vanishing first minors of $\lambda - \mathbf{a}_1$. Evidently the first minors with respect to the elements above the principal diagonal vanish, since all the elements below the principal diagonal of such a minor, and at least one element in the principal diagonal, are zero. The minor with respect to an element in the principal diagonal has the value $(\lambda-\lambda_1)^{m-1}$, where m is the order of $\mathbf{a}_1$. If in forming a first minor with respect to an element below the principal diagonal, one of the zeros that separate the sets of consecutive 1's is inside the right angle formed by the row and column that are deleted to form the first minor in question, then that zero is introduced into the principal diagonal of the first minor, which, therefore, vanishes. Consequently, the only elements of $\mathbf{a}_1$ whose minors do not vanish are those whose rows and columns contain only consecutive 1's from the over-principal diagonal of $\mathbf{a}_1$ inside the right angle formed by those rows and columns. Consequently the minimum degree of any non-vanishing first minor in $\lambda-\lambda_1$ is $m-e_1$. Similarly the minimum degree of $\lambda-\lambda_1$ in the 2nd, 3rd, ... minors is $m-(e_1+e_2)$, $m-(e_1+e_2+e_3)$, ... Evidently the first difference of these numbers are $e_1, e_2, e_3, \ldots$; that is, the e's are the exponents of the elementary divisors with respect to the root λ_1. In a precisely similar manner we could find corresponding numbers for the other roots of the characteristic equation of the original matrix $\mathbf{a}$. The e's of our normal form are, therefore, the exponents of the elementary divisors of the matrix.

We have, as the result of the paper up to the present, the

THEOREM. The necessary and sufficient condition that the two non-singular families of bilinear forms

$$\lambda' A+\lambda'' B, \tag{1}$$

$$\lambda' A'+\lambda'' B' \tag{2}$$

are equivalent, is that the matrices corresponding to the forms

$$\lambda-AB^{-1},$$

$$\lambda-A'B'^{-1}$$

have, as their complete system of invariants under the similarity transformation, (I) their order, (II) the roots of their characteristic equations, (III) the exponents of their elementary divisors.

These three invariants may be included in one invariant; namely, the elementary divisors of the matrices of these forms.

§ 5.

We proceed to show that the necessary and sufficient condition that the two non-singular families $\lambda' A+\lambda'' B$ and $\lambda' A'+\lambda'' B'$ are equivalent is that their own elementary divisors are identical. The necessity of this condition is shown in the same way as the invariance of the elementary divisors of a matrix was shown at the end of § 2, and will not be repeated here.*

It also appears immediately that if the matrices $\mathfrak{ab}^{-1}$ and $\mathfrak{a}'\mathfrak{b}'^{-1}$ have identical elementary divisors, then (1) and (2) also have identical elementary divisors. For then $\mathfrak{ab}^{-1}$ and $\mathfrak{a}'\mathfrak{b}'^{-1}$ are similar, hence (1) and (2) are equivalent, and have the same elementary divisors. It only remains to show that if (1) and (2) have equal elementary divisors, then $\mathfrak{ab}^{-1}$ and $\mathfrak{a}'\mathfrak{b}'^{-1}$ also have equal elementary divisors.

Let $\mathfrak{ab}^{-1} = \mathfrak{n}$, and consider the family $\lambda_1\mathfrak{n} + \lambda_2$. We assume that $\mathfrak{n}$ is in normal form. This family is equivalent to (1) by the criterion derived in § 1.

Now if the matrix of this family $\lambda_1\mathfrak{n} + \lambda_2$ is written out, it is observed that the principal diagonal consists of elements of the form $\lambda_1\lambda_i + \lambda_2$, where the λ_i are roots of the characteristic equation of the matrix $\mathfrak{ab}^{-1}$, and the over-principal diagonal consists of unit elements or zeros, in the same order as they occur in the over-principal diagonal of the matrix $\mathfrak{n}$. That is to say, we have here a form of the matrix of the family of (1) that displays its elementary divisors in just the same way as the elementary divisors of $\mathfrak{ab}^{-1}$ are displayed in $\mathfrak{n}$. It is, in fact, a normal form for (1), and it appears that the exponents of the elementary divisors for $\mathfrak{n}$ are the same as those for (1). Consequently, when we find the normal form for $\mathfrak{ab}^{-1}$, we are simultaneously finding the normal form (1), and if (1) is known to have certain elementary divisors, we can write down without further discussion the elementary divisors of $\mathfrak{ab}^{-1}$, for their exponents are identical. Consequently, if (1) and (2) have the same elementary divisors, $\mathfrak{ab}^{-1}$ and $\mathfrak{a}'\mathfrak{b}'^{-1}$ also have the same elementary divisors, and (1) and (2) are therefore equivalent. This is Weierstrass's great theorem.

Practically to determine whether two families of bilinear forms of the types (1) and (2) are equivalent, we must build the matrices $\mathfrak{ab}^{-1}$ and $\mathfrak{a}'\mathfrak{b}'^{-1}$, and reduce both of these matrices to their normal form. If they have identical (I) rank, (II) characteristic equations, (III) exponents of their elementary divisors, then (1) and (2) are equivalent. It is to be observed that if invariants (I) or (II) of the two matrices differ, the fact is always noticed before any reduction is made, and the fact of inequivalence is shown without the labor of reduction.

YALE UNIVERSITY, *December* 11, 1908.

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*For a detailed proof of this fact see Bôcher, Introduction to Higher Algebra.

Basic Systems of Rational Norm-Curves.

By J. R. Conner.

I.

Introductory.

§ 1. *Basic Systems Defined.*

The *base,* or $(n+2)$-point in a space of n dimensions, derives its importance from the fact that it is the figure of the greatest number of independent points that is projectively equivalent to any other such set. The assumption of two ordered bases in two separate or superposed spaces gives a collineation between these spaces.

We shall use the symbol S_n in referring to a flat space of n dimensions, and shall designate the rational norm-curve in n dimensions by the symbol R^n. An R^n has n^2+2n-3 constants. It is $n-1$ conditions on a curve in n dimensions to pass through a given point, and hence R^n's on $n+2$ points have

$$n^2+2n-3-(n-1)(n+2)=n-1$$

degrees of freedom. We shall call a system of ∞^{n-1} R^n's on $n+2$ points in n dimensions a *basic system of rational norm-curves,* and individual curves of such a system, basic R^n's. There is a unique R^n on $n+3$ points, in general, and hence through the general point of the S_n under consideration passes one and only one basic R^n.

§ 2. *Contact of Basic R^n's with Spreads in S_n.*

The following theorem is one of which use will be made throughout this paper. It is inserted here as a matter of convenience.

* *Basic R^n's in S_n having p-point contact with an $(n-1)$-way spread,*

$$(\alpha x)^m = 0,$$

of order m, touch at points of an $(n-p)$-way spread of order

$$m(m+1)(m+2)\ldots\ldots(m+p-1),$$

which is the complete intersection with α of $(n-1)$-way spreads of orders

$$m+1, \quad m+2, \quad \ldots\ldots, \quad m+p-1.$$

Proof: Take as base the $n+1$ points of the reference figure and the unit point. An R^n on these points and a point x is in general uniquely determined, and may be given parametrically by

$$\rho x_i = \frac{x_i}{1 - x_i t} \quad (i = 1, \ldots, n+1). \tag{1}$$

$t = \frac{1}{x_i}$ gives the points of reference, $t = \infty$ gives the unit point, and $t = 0$ gives the point x.

(1) may also be written

$$\sigma x_i = \frac{x_i \prod\limits_{j=1}^{n+1}(1 - x_j t)}{1 - x_i t},$$

or

$$\sigma x_i = x_i\,[1 - s_{1,i}\,t + s_{2,i}\,t^2 - s_{3,i}\,t^3 + \ldots\ldots + (-)^n s_{n,i}\,t^n], \tag{2}$$

where $s_{j,i}$ is the sum of products j at a time of all the x's but x_i. The R^n (2) meets $(\alpha x)^m = 0$ in the mn points given by

$$[(\alpha x) - (\alpha x s_1)\,t + (\alpha x s_2)\,t^3 - \ldots\ldots]^m = 0, \tag{3}$$

where

$$(\alpha x) \equiv \alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots,$$
$$(\alpha x s_j) \equiv \alpha_1 x_1 s_{j,1} + \alpha_2 x_2 s_{j,2} + \ldots\ldots.$$

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* Compare Humbert: "Sur un Complexe Remarquable de Coniqués," *Journal de l'École Polytechnique*, Cah. 64 (1894), p. 125. Humbert's argument that basic R^3's touch an n-ic surface at points of a curve of order $2n^2$ and osculate in $6n^3$ points is obviously incorrect. His equations (6) should be

$$f(x, y, z, t) = 0,$$
$$x^2\frac{\partial f}{\partial x} + y^2\frac{\partial f}{\partial y} + z^2\frac{\partial f}{\partial z} + t^2\frac{\partial f}{\partial t} = 0,$$

which is the form given further on.

Expanding (3) and arranging in ascending powers of t, we have

$$\left.\begin{aligned}&(\alpha x)^m - m\,(\alpha x)^{m-1}(\alpha x s_1)\,t + \left[\tbinom{m}{2}(\alpha x)^{m-2}(\alpha x s_1)^2 + m\,(\alpha x)^{m-1}(\alpha x s_2)\right]t^2\\ &\quad -\left[\tbinom{m}{3}(\alpha x)^{m-3}(\alpha x s_1)^3 + 2\tbinom{m}{2}(\alpha x)^{m-2}(\alpha x s_1)(\alpha x s_2) + m\,(\alpha x)^{m-1}(\alpha x s_3)\right]t^3\\ &\quad +\left[\tbinom{m}{4}(\alpha x)^{m-4}(\alpha x s_1)^4 + 3\tbinom{m}{3}(\alpha x)^{m-3}(\alpha x s_1)^2(\alpha x s_2) + \tbinom{m}{2}(\alpha x)^{m-2}(\alpha x s_2)^2\right.\\ &\quad \left.+\,2\tbinom{m}{2}(\alpha x)^{m-2}(\alpha x s_1)(\alpha x s_3) + m\,(\alpha x)^{m-1}(\alpha x s_4)\right]t^4 - \ldots\ldots = 0.\end{aligned}\right\}\quad(4)$$

If x is a point of $(\alpha x)^m = 0$, t is a factor of (4), and the remaining roots of (4) are the parameters of points of intersection of (2) with α. If we desire the basic R^n on x to touch the spread α at x, t^2 must be a factor of (4) and we must have

$$(\alpha x)^m = 0,$$
$$(\alpha x)^{m-1}(\alpha x s_1) = 0.$$

If we desire p-point contact, t^p must be a factor of (4); the coefficients of $1, t, t^2, \ldots, t^{p-1}$ in (4) must vanish. But these represent $(n-1)$-way spreads of orders

$$m, \quad m+1, \quad m+2, \quad \ldots, \quad m+p-1,$$

respectively. Hence the theorem.

The coefficients of (4) may be modified by the identities

$$\left.\begin{aligned}&(\alpha x s_1) \equiv (\alpha x)\,\sigma_1 - (\alpha x^2),\\ &(\alpha x s_2) \equiv (\alpha x)\,\sigma_2 - (\alpha x^2)\,\sigma_1 + (\alpha x^3),\\ &(\alpha x s_3) \equiv (\alpha x)\,\sigma_3 - (\alpha x^2)\,\sigma_2 + (\alpha x^3)\,\sigma_1 - (\alpha x^4),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\end{aligned}\right\}\quad(5)$$

where the σ's are symmetric functions of all the x's, and

$$(\alpha x^j) \equiv \alpha_1 x_1^j + \alpha_2 x_2^j + \ldots\ldots$$

Using the identities (5), we have from (4):

The conditions for p-point contact of a basic R^n with $(\alpha x)^m = 0$ at the point x are

$$\left.\begin{aligned}&(\alpha x)^m = 0,\\ &(\alpha x)^{m-1}(\alpha x^2) = 0,\\ &(m-1)(\alpha x)^{m-2}(\alpha x^2)^2 + 2\,(\alpha x)^{m-1}(\alpha x^3) = 0,\\ &(m-1)(m-2)(\alpha x)^{m-3}(\alpha x^2)^3 + 6\,(m-1)(\alpha x)^{m-2}(\alpha x^2)(\alpha x^3)\\ &\qquad\qquad + 6\,(\alpha x)^{m-1}(\alpha x^4) = 0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\}\quad(6)$$

The forms (6) may be obtained more directly by substitution of the coördinates of x as given by (1) in $(\alpha x)^m$, differentiating $p-1$ times as to t and putting $t=0$

in the resulting expressions.* The spreads (6) behave very much like the successive polar spreads of α, and the modifications of the theorem for the case where α has multiple spreads are obvious. A specially interesting case arises when basic R^n's cannot have p-point contact with α without lying entirely on it; we shall see several applications of the theorem in this connection.

We have from (6):

Basic R^n's have p-point contact with an S_{n-1}, $(\alpha x) = 0$, where

$$(\alpha x) = (\alpha x^2) = (\alpha x^3) = \ldots . = (\alpha x^p) = 0. \tag{7}$$

II.

BASIC NORM-CURVES IN TWO DIMENSIONS.

§ 3. *Applications of the Theorem of Part I.*

The base in a plane is a set of four points. The basic system of norm-curves is here the pencil of conics on the four points. These conics define an involution on every line of the plane. The double points of this involution are the points of contact of the two basic conics that touch the line. The points of contact of basic conics with a given line, $(\alpha x) = 0$, are cut out by

$$(\alpha x^2) = 0.$$

Basic conics touch an m-ic curve

$$\alpha \equiv (\alpha x)^m = 0 \tag{1}$$

in the $m(m+1)$ points cut out by

$$(\alpha x)^{m-1}(\alpha x^2) = 0. \tag{2}$$

If (1) is on a base-point, (2) touches (1) there. If (1) is on a point of the diagonal triangle of the base, (2) passes through that point, but does not in general touch (1) there. More generally, if (1) has an i-fold point at a base-point, (2) has an i-fold point there, with the same tangents. If (1) has an i-fold point at a point of the diagonal triangle of the base, (2) has also an i-fold point there, but not with the same tangents in general.

(2) vanishes if the conditions for a node of α are satisfied; that is, if

$$(\alpha x)^{m-1}\alpha_i = 0 \quad (i = 1, 2, 3).$$

Hence (2) passes through all the nodes of α, and two is to be deducted from the number of actual contacts for each of the nodes.

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*Humbert, *loc. cit.*

If α have a cusp at x, we must have

$$\frac{\partial\alpha}{\partial x_1}=\frac{\partial\alpha}{\partial x_2}=\frac{\partial\alpha}{\partial x_3}=0,$$

and

$$y_1^2\frac{\partial^2\alpha}{\partial x_1^2}+y_2^2\frac{\partial^2\alpha}{\partial x_2^2}+y_3^2\frac{\partial^2\alpha}{\partial x_3^2}+2y_2y_3\frac{\partial^2\alpha}{\partial x_2\partial x_3}+\cdots$$

is a perfect square. This last expression, if a perfect square, is the square of

$$y_1\sqrt{\frac{\partial^2\alpha}{\partial x_1^2}}+y_2\sqrt{\frac{\partial^2\alpha}{\partial x_2^2}}+y_3\sqrt{\frac{\partial^2\alpha}{\partial x_3^2}}=0, \tag{3}$$

the square roots being those definite roots determined by the conditions

$$\sqrt{\frac{\partial^2\alpha}{\partial x_i^2}}\sqrt{\frac{\partial^2\alpha}{\partial x_j^2}}=\frac{\partial^2\alpha}{\partial x_i\partial x_j}. \tag{4}$$

(3) equated to zero is the cusp tangent to α at x. Now the tangent to (2) at x is, dropping the terms in $\frac{\partial\alpha}{\partial x_i}$,

$$\left.\begin{aligned}&y_1\left(x_1^2\frac{\partial^2\alpha}{\partial x_1^2}+x_2^2\frac{\partial^2\alpha}{\partial x_1\partial x_2}+x_3^2\frac{\partial^2\alpha}{\partial x_1\partial x_3}\right)\\&\quad+y_2\left(x_1^2\frac{\partial^2\alpha}{\partial x_1\partial x_2}+x_2^2\frac{\partial^2\alpha}{\partial x_2^2}+x_3^2\frac{\partial^2\alpha}{\partial x_2\partial x_3}\right)\\&\quad+y_3\left(x_1^2\frac{\partial^2\alpha}{\partial x_1\partial x_3}+x_2^2\frac{\partial^2\alpha}{\partial x_2\partial x_3}+x_3^2\frac{\partial^2\alpha}{\partial x_3^2}\right)=0.\end{aligned}\right\} \tag{5}$$

(5), on account of the conditions (4), may be written

$$\left(y_1\sqrt{\frac{\partial^2\alpha}{\partial x_1^2}}+y_2\sqrt{\frac{\partial^2\alpha}{\partial x_2^2}}+y_3\sqrt{\frac{\partial^2\alpha}{\partial x_3^2}}\right)\left(x_1^2\sqrt{\frac{\partial^2\alpha}{\partial x_1^2}}+x_2^2\sqrt{\frac{\partial^2\alpha}{\partial x_2^2}}+x_3^2\sqrt{\frac{\partial^2\alpha}{\partial x_3^2}}\right)=0.$$

Hence (2) passes through all cusps of (1) and touches the cusp-tangents at the cusps. We have, then,

If an m-ic curve, not related to the base, have δ nodes and $\varkappa$ cusps, the number of basic conics having proper contact with it is

$$N=m(m+1)-2\delta-3\varkappa.$$

We can now find the order of the curve of contact of basic conics with a pencil of m-ics. An m-ic of the pencil can meet the curve only at its points of contact with basic conics, since there is only one m-ic of the pencil on a general point of the plane, and at the base-points of the pencil of m-ics. We have, then, as the number of intersections of an m-ic with the curve,

$$m^2 + m(m+1) = m(2m+1).$$

This gives the theorem:

* *The curve of contact of a pencil of conics and a pencil of m-ics is a curve of order* $2m+1$.

If we have a pencil of m-ics

$$(\alpha x)^m + \lambda(\beta x)^m = 0,$$

the contacts of basic conics with the curves of this pencil are cut out by the corresponding curves of the pencil

$$(\alpha x)^{m-1}(\alpha x^2) + \lambda(\beta x)^{m-1}(\beta x^2) = 0.$$

Eliminating λ between these two equations, we have

$$\begin{vmatrix} (\alpha x)^m & (\beta x)^m \\ (\alpha x)^{m-1}(\alpha x^2) & (\beta x)^{m-1}(\beta x^2) \end{vmatrix} = 0 \qquad (6)$$

as the equation of the curve of contact. If $(\alpha x)^m$ have a node at a point y, $(\alpha x)^{m-1}(\alpha x^2)$ passes through y, and y is a simple point of (6). Similarly, if $(\alpha x)^m$ have a p-fold point at y, y is a $(p-1)$-fold point of (6). (6) obviously passes through the base-points of the pencil of m-ics. It is also on the four points of the base, and the three points of their diagonal triangle.

§ 4. *A Rational Sextic Associated with the Base.*

Certain curves that arise in connection with systems of basic conics in a plane are of importance in the discussion of higher dimensional cases. We mention some of these here, though the discussion of the special cases of these curves that we shall meet later will be easier in the light of facts that have not yet been brought out.

A sextic curve, Σ, is determined by the basic conics and a general conic, C, in the following manner: Through a point of intersection of a basic conic with C draw the tangent to C. *The locus of the point where this tangent meets the basic conic again is a rational sextic curve.*

For it obviously has nodes at each base-point, the double point being determined by the two basic conics on the two points of tangency of tangents from the base-point to C. Further, a basic conic, apart from intersections at the base-points, can meet the curve only in the four points determined by its intersections with C, since there is one and only one basic conic through a point of the curve, and this must be its defining conic. We see, then, that a basic conic has $4 + 2 . 4 = 12$ intersections with Σ, and hence it is of order 6, being met by a conic in twelve points. Σ is in one-one correspondence with C and hence must be rational. It must touch C at the six points of contact of basic conics with C. For these basic conics obviously touch Σ there.

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* This is a special case of a more general theorem. See Pascal: "Rep. der Höheren Math.," II, p. 482.

Four of the ten nodes of Σ are at the base-points. Since there is only one basic conic on a given point of the plane other than a base-point, the only way in which additional nodes can arise is for the pole as to C of a common chord of C and a basic conic ϕ to be on ϕ. If this happens for a given common chord of C and ϕ it will happen for the opposite common chord. Hence:

Six nodes of Σ lie by twos on three basic conics. The other four nodes are at the base-points.

A rational sextic in a plane has seventeen constants. Σ, being determined by a conic and a base, has $5 + 8 = 13$ constants. Three conditions on Σ are accounted for by the theorem as to the nodes. The additional condition seems to be the existence of a *perspective conic*, C.[*]

§ 5. *A Point-Sextic and a Line-Quartic Determined by Two Bases.*

The base in a plane determines a quadratic Cremona involution x, y such that x and y are apolar to all basic conics. The polar lines of x with regard to basic conics are on y. x and y are the points of contact of basic conics with the line xy. The base-points are fixed points of this transformation and the three points of the diagonal triangle are singular points. A line is carried by the transformation into a conic on the singular points. A conic goes into a rational quartic with nodes at the singular points.

There are four pairs of corresponding points on every conic of the plane, namely, the eight points in which the conic meets its correspondent quartic. Assuming the basic conics as given, a pencil of conics Φ determines a curve of points x whose correspondent, y, as to the basic conics, lies on the same conic Φ with x. x is subject to the condition that the points x, y, determined by the basic conics, shall be apolar to the points x', y', the points of contact of conics Φ with the line xy. The locus of x, S say, is evidently on the base-points of Φ, since the involution x', y' is parabolic at these base-points. If x is a base-point of Φ, the point y determined by x will, then, satisfy the condition of apolarity to $x'y'$. A conic Φ meets S in $8+4=12$ points. Hence:

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[*] Stahl: "Zur Erzeugung der ebenen rat. Curven," *Math. Ann.*, XXXVIII. Two rational curves, one given in lines and the other in points, are in ***perspective position*** when they are in one-one correspondence and corresponding lines and points are incident. This requires, in general, that the two curves shall touch, or that the point-curve shall have a cusp, at every common corresponding point. For, let us say that the line-curve Λ meets the point-curve L at a common corresponding point a. Consider L as generated by a point of a line of Λ, suitably fixed. This point approaches a as the point of tangency of the tangent to Λ approaches a. But a variable point on a tangent to a curve, at an ordinary point and in the neighborhood of the point of tangency, cannot cross the curve. Hence our statement. It is an easy inference from Stahl's theory that a point-curve and a line-curve in perspective position must, in general, have $m + n - 2$ common corresponding points, m and n being the order and class respectively of the two curves.

The locus of a point, x, whose correspondent, y, in the Cremona involution determined by the basic conics, lies on a conic with x and four fixed points of the plane, is a sextic curve S.

This sextic is invariant under the quadratic involution, corresponding points on it being interchanged. It follows that it must have nodes at the diagonal points of the base.

These facts may by reached analytically. Let us assume as base the points $(1, \pm 1, \pm 1)$ with the reference points as diagonal points. The Cremona involution determined by basic conics is now simply

$$\rho y_i = \frac{1}{x_i}. \tag{7}$$

Suppose the pencil of conics, Φ, to be

$$\Phi \equiv (\alpha x)^2 + \lambda (\beta x)^2 = 0. \tag{8}$$

The points x whose corresponding points lie on the same conic Φ with x must satisfy in addition, according to (7),

$$(\alpha/x)^2 + \lambda (\beta/x)^2 = 0, \tag{9}$$

where

$$(\alpha/x) \equiv \alpha_1/x_1 + \alpha_2/x_2 + \alpha_3/x_3.$$

Eliminating λ between (8) and (9), we have

$$S \equiv \begin{vmatrix} (\alpha x)^2 & (\beta x)^2 \\ (\alpha/x)^2 & (\beta/x)^2 \end{vmatrix} = 0, \tag{10}$$

as the equation of our sextic. $(\alpha/x)^2$ and $(\beta/x)^2$ are rational quartics with nodes at the diagonal points of the base. The form of (10) gives us with regard to the sextic S:

1) S is invariant under the involution (7).
2) S has nodes at the diagonal points of the base.
3) S is on the base-points of both pencils of conics.

The locus of points of contact of basic conics with the lines of a pencil is a curve of order $2.1 + 1 = 3$, as is otherwise known. This cubic is on the four base-points and the three diagonal points. It is evidently invariant under (7), points of contact on each line being interchanged. Every point of the plane determines a cubic of this sort; there are ∞^2 of them and they form a linear system. A cubic determined in this way by a point a meets S in $18 - 3.2 - 4 = 8$ points, apart from the seven. This says that through any point a of the plane there are four lines joining pairs of corresponding points on the sextic S. Hence:

Lines joining corresponding points of the sextic S touch a curve of class four.

The equation of this quartic may be written down at once by means of the Clebsch translation-principle. Given two pencils of binary quadratics

$$(\alpha x)^2 + \lambda (\beta x)^2 = 0,$$
$$(\gamma x)^2 + \mu (\delta x)^2 = 0,$$

the apolarity-condition of the Jacobians of the two pencils is

$$|\alpha\beta| \; |\gamma\delta| \; [|\alpha\gamma| \; |\beta\delta| + |\alpha\delta| \; |\beta\gamma|] = 0,$$

or

$$|\alpha\gamma|^2 \, |\beta\delta|^2 - |\alpha\delta|^2 \, |\beta\gamma|^2 = 0, \tag{11}$$

since

$$|\alpha\beta| \; |\gamma\delta| \equiv |\alpha\gamma| \; |\beta\delta| - |\alpha\delta| \; |\beta\gamma|.$$

The equation of our quartic is, from (11),

$$|\alpha\gamma\xi|^2 \, |\beta\delta\xi|^2 - |\alpha\delta\xi|^2 \, |\beta\gamma\xi|^2 = 0, \tag{12}$$

this being the equation of the quartic determined by the two pencils of conics

$$(\alpha x)^2 + \lambda (\beta x)^2 = 0,$$
$$(\gamma x)^2 + \mu (\delta x)^2 = 0.$$

If $(\gamma x)^2 + \mu (\delta x)^2$ is the pencil of conics on the base $(1, \pm 1, \pm 1)$, we have

$$\gamma_{ij} = \delta_{ij} = \Sigma \gamma_{ii} = \Sigma \delta_{ii} = 0,$$

and (12) becomes,

$$\left.\begin{aligned} &\Sigma (\alpha_{22}\beta_{33} - \alpha_{33}\beta_{22}) \xi_1^4 + 2\Sigma (\alpha_{11}\beta_{23} - \alpha_{23}\beta_{11}) \xi_2 \xi_3 (\xi_2^2 - \xi_3^2) \\ &\quad + \Sigma [\alpha_{11}(\beta_{33} - \beta_{22}) - \beta_{11}(\alpha_{33} - \alpha_{22})] \xi_2^2 \xi_3^2 \\ &\quad + 2\Sigma [\alpha_{23}(\beta_{33} - \beta_{22}) - \beta_{23}(\alpha_{33} - \alpha_{22}) + 2(\alpha_{12}\beta_{13} - \beta_{12}\alpha_{13})] \xi_1^2 \xi_2 \xi_3 = 0. \end{aligned}\right\} \tag{13}$$

The terms not given may be obtained by cyclic permutation of the subscripts.

The equation (13) shows that the quartic touches the six lines $(1, \pm 1, 0)$, $(1, 0, \pm 1)$, $(0, 1, \pm 1)$, the six lines joining the base-points two and two. By symmetry we infer that it also touches the six lines joining the base-point of Φ.

It will be observed that two sextics are determined by the basic conics and the pencil Φ, one, S, whose equation we have found above, and another, S', the locus of pairs of corresponding points x', y' which lie on basic conics. The equation of S' may be obtained as we found that of S. It is, in determinant form,

$$S' \equiv \begin{vmatrix} 1, & 1, & 1 \\ x_1^2, & x_2^2, & x_3^2 \\ y_1^2, & y_2^2, & y_3^2 \end{vmatrix} = 0, \tag{14}$$

where

$$y_i = \frac{\partial}{\partial a_i} |a\,\alpha\,\beta|\,(\alpha x)\,(\beta x), \tag{15}$$

(15) being the Cremona involution determined by the conics Φ. The quartic (13) is symmetrically related to both pencils of conics and to both sextics. Its lines meet the two sextics in harmonic pairs.

III.

BASIC NORM-CURVES IN THREE DIMENSIONS.

§ 6. *Introductory.*

The base in three dimensions is a set of five points. There is a system of ∞^3 norm-curves (R^3's) on the five points, since an R^3 has twelve coördinates, and it is ten conditions on a curve in space to pass through five given points. In addition we have a linear system of ∞^4 quadric surfaces on the five base-points, which we shall call *basic quadrics*. In our discussion of the subject we shall make use of the following known theorems:

a) *Basic quadrics meet any plane* α *in conics apolar to a definite conic* C_α *in* α.[*]

b) *Basic* R^3*'s meet* α *in sets of three points apolar to* C_α.[†]

These sets of three points are very important in the consideration of the apparatus determined by basic R^3's on a plane. We shall use the symbol $\iota_3^{(3)}$ in referring to such a set of points. In general, we shall use the symbol $\iota_r^{(n)}$ in referring to a set of r points of a basic R^n on any flat under consideration.

c) *Basic* R^3*'s tangent to* α *touch at points of* C_α.[‡]

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[*] Compare Loria, *R. Istituto Lombardo*, April, 1884.

[†] Reye: "Geometrie der Lage," Part II, pp. 223, ff.

[‡] Reye: *loc. cit.*

d) *Six basic R^3's osculate α, the points of osculation being on C_α.*[*]

e) *Basic R^3's tangent to α meet α again in points of a rational sextic curve with nodes at the ten points of the Desargues configuration (configuration B) determined on α by the five base-points.*[*]

f) *Basic elliptic quartics (E^4's) meet α in sets of four points orthic to C_α.*[†]

By a set of four points *orthic* to C_α we mean four points such that any conic on them taken in points is apolar to C_α taken in lines. Dualistically we have sets of four lines orthic to C_α.

C_α may be defined as the conic with regard to which the configuration B determined on α by the base is self-polar. We name the points of the base 1, 2, 3, 4, 5, the ten lines joining two of these five points 12, etc., and the ten planes joining three of them 123, etc., and name the points and lines of the configuration B on α after the lines and planes of the base by which they are determined.

§ 7. *The Point-Sets $\iota_3^{(3)}$.*

Consider the whole set of basic R^3's as projected from a base-point, say 5. This gives us a pencil of quadric cones with vertex at 5 and on the four lines 15, 25, 35, 45. On each of these cones there is a single infinity of basic R^3's, one R^3 through every point of the cone. A basic R^3 through a point a of one of these cones must lie entirely on the cone, since it has seven points in common with it, four at 1, 2, 3, 4, two at 5, and one at a.

This pencil of quadric cones meets α in a pencil of conics on 15, 25, 35, 45, all of which are apolar to C_α by theorem a). 15, 25, 35, 45 are an orthic 4-point as to C_α. A defining characteristic of such a set of points is that the pole of the join of any two is on the opposite line of the complete 4-point. A conic of this pencil, being apolar to C_α, must contain a single infinity of 3-points apolar to C_α. Since one point not a base-point and not on a base-line is sufficient to determine uniquely a basic R^3, and since one point of a conic of the pencil is sufficient to determine uniquely a 3-point apolar to C_α, we have the theorem:

3-points $\iota_3^{(3)}$ on α are 3-points inscribed in conics of the pencil 15, 25, 35, 45, *and apolar to C_α, and conversely.*

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[*] Sturm, *Crelle's Journal*, Vol. LXXIX, p. 99. There are many points of contact between this paper of Sturm and the paper of Humbert cited earlier, on the one hand, and the present section of this paper on the other, but the point of view is somewhat different.

[†] This in an easy consequence of theorem a).

In the configuration B there are five orthic 4-points, namely:

15	25	35	45	(5)
14	24	34	54	(4)
13	23	43	53	(3)
12	32	42	52	(2)
21	31	41	51	(1)

These may be called the sets 5, 4, 3, 2, 1 respectively. Two sets i and j have one point and only one in common—the point ij. i-conics and j-conics can, then, meet in only three variable points, and these sets of points are always $\iota_3^{(3)}$'s. We have the theorem:

The three points of a set $\iota_3^{(3)}$ *are on a conic with any orthic 4-point of* B.

A set of points $\iota_3^{(3)}$ may be constructed when one of the three points is given by drawing the conic on 15, 25, 35, 45, and the given point. The other two points are the intersections with this conic of the polar line of the given point as to C_a. This gives us at once theorem c) above, and the following:

Basic R^3*'s osculate* a *at the six points of contact with* C_a *of conics on* 15, 25, 35, 45.

Conics of all orthic pencils of B *touch* C_a *at the same six points.**

The sextic of theorem e) *is the locus of the point in which the tangent to* C_a, *at a point of a conic of a given orthic pencil of* B, *meets this latter conic again.*

§ 8. *The Sextic of Theorem* e).

Let us examine this sextic a little more closely. We saw in Part II that a sextic generated in this way has nodes at the base-points of the generating pencil, and touches C_a at the six points of contact of conics of that pencil. Since the pole of each line of an orthic 4-point is on the opposite line, and a line taken with its opposite line is a degenerate conic of the pencil, the six remaining nodes of the sextic are at the poles of the lines of our orthic 4-point, and we have theorem e).

Considering 15, 25, 35, 45 as a *base* in a, the equation of this sextic may be obtained by demanding that the polar line of a point x as to C_a touch the basic conic on x. Choosing as reference points 15, 25, 35, with 45 as unit point, a basic conic may be written:

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* This may be regarded as a geometrical determination of the five quartic involutions associated with a given sextic as Jacobian. Cf. Meyer, "Apolarität," pp. 305, ff. B is unique when the sextic is given.

$$m_1 x_2 x_3 + m_2 x_3 x_1 + m_3 x_1 x_2 = 0, \tag{1}$$

with the condition

$$m_1 + m_2 + m_3 = 0. \tag{2}$$

C_α must be of the form

$$a_1 \xi_1^2 + a_2 \xi_2^2 + a_3 \xi_3^2 + 2\lambda(\xi_2 \xi_3 + \xi_3 \xi_1 + \xi_1 \xi_2) = 0, \tag{3}$$

the apolarity-condition of (1) and (3) being merely

$$\lambda(m_1 + m_2 + m_3),$$

which vanishes by (2). The point-equation of (3) is

$$(cx)^2 \equiv (a_2 a_3 - \lambda^2) x_1^2 + \dots + 2(\lambda^2 - \lambda a_1) x_2 x_3 + \dots = 0. \tag{4}$$

The basic conic on x is, parametrically,

$$y_i = \frac{x_i}{1 - x_i t} \qquad (i = 1, 2, 3). \tag{5}$$

The polar line of x as to $(cx)^2 \equiv C$ is

$$y_1 \frac{\partial C}{\partial x_1} + y_2 \frac{\partial C}{\partial x_2} + y_3 \frac{\partial C}{\partial x_3} = 0, \tag{6}$$

Substituting (5) in (6), we have the quadratic in t,

$$\sum^{123} \frac{\partial C}{\partial x_1} x_1 (1 - x_2 t)(1 - x_3 t) = 0,$$

which, arranged in powers of t, is

$$x_1 x_2 x_3 \Sigma \frac{\partial C}{\partial x_1} t^2 - \Sigma x_1 x_2 \left(\frac{\partial C}{\partial x_1} + \frac{\partial C}{\partial x_2} \right) t + 2C = 0, \tag{7}$$

since

$$\Sigma x_1 \frac{\partial C}{\partial x_1} \equiv 2C.$$

The condition that (6) touch (5) is the discriminant of (7). The equation of the rational sextic is, then,

$$8 x_1 x_2 x_3 \Sigma \frac{\partial C}{\partial x_1} C - \left[\Sigma x_1 x_2 \left(\frac{\partial C}{\partial x_1} + \frac{\partial C}{\partial x_2} \right) \right]^2 = 0.$$

This is of the form

$$8 x_1 x_2 x_3 L C - f_3^2 = 0, \tag{8}$$

where f_3 is a cubic, and L is the polar line of the unit point (45) as to C_α, a configuration line. (8) obviously touches C_α at its intersections with f_3. Hence f_3 is a cubic on the six points of osculation of basic R^3's with α.

The four points 15, 25, 35, 45, and the conic C_α are sufficient to determine the configuration B on α. The points of B are the four base-points and the poles as to C_α of the six lines joining them. The lines of B are these six lines and the polar lines of the base-points. The coördinates of the points of B are:

$$\left.\begin{array}{llllll} 15 & (1,\ 0,\ 0), & 14 & (a_1,\ \lambda,\ \lambda), & 23 & [0,\ a_2-\lambda,\ -(a_3-\lambda)], \\ 25 & (0,\ 1,\ 0), & 24 & (\lambda,\ a_2,\ \lambda), & 31 & [-(a_1-\lambda),\ 0,\ a_3-\lambda], \\ 35 & (0,\ 0,\ 1), & 34 & (\lambda,\ \lambda,\ a_3), & 12 & [a_1-\lambda,\ -(a_2-\lambda),\ 0], \\ 45 & (1,\ 1,\ 1). & & & & \end{array}\right\} \qquad (9)$$

We give in this table the names of the points in two-figure symbols.

(8) shows that the cubic f_3 meets the lines $x_1\, x_2\, x_3\, L$ at nodes of the sextic. This gives the theorem:

The six points of contact of basic conics with C, or the six points of osculation of basic R^3's with α, and the six points 15, 25, 35, 12, 23, 31 *are on a cubic curve.*

f_3 is, in full,

$$\tfrac{1}{2} f_3 \equiv \sum^{123} a_3 (a_2-\lambda)\, x_1^2 x_2 + [6\lambda^3 - 2\lambda(a_1+a_2+a_3)]\, x_1 x_2 x_3 = 0. \qquad (10)$$

(10) is, in terms of the cubic,

$$(c\,x)(c\,x^2) = 0,$$

which, as we have found, cuts out the contacts of basic conics with C_α,

$$\tfrac{1}{2} f_3 \equiv (x_1+x_2+x_3)\,C - (c\,x)(c\,x^2). \qquad (11)$$

The above theorem is immediately evident from (10) and (11) by actual substitution of the coördinates of the six points as given in the table (9). These six points are the vertices of the 4-line of B, 235, 135, 125, 123. This is the polar reciprocal as to C_α of the orthic 4-point 4, and is an orthic 4-line as to C_α.

There can be but one sextic with nodes at the ten points of a configuration B; two would have forty points of intersection. Although our method of generation is unsymmetrical, the sextic is symmetrical with regard to the configuration and may equally well be generated by any one of the other four orthic pencils of B. Again, the whole figure is self-dual, and any theorem referring to the configuration carries with it its reciprocal theorem; polar reciprocation in C_α is a sufficient proof for this statement. We have then:

The six vertices of any orthic 4-line of B and the six points of contact with C_α of conics of the range touching these lines, lie on a curve of order 3.

We have in this way five cubics. Any two of these cubics intersect only in the six points on C_α and in the three points of a configuration line. Also:

The six lines of any orthic 4-point of B and the six lines of contact with C_α of conics on the 4 points, touch a curve of class 3.[*]

A metrically special, but projectively general, rational sextic with nodes at the ten points of a configuration B may easily be constructed by choosing 15, 25, as the circular points in the plane; 35, 45, are then inverse points in the director circle of C_α, and the pencil of conics on 15, 25, 35, 45 is then a pencil of circles on two points.[†]

§ 9. *Applications of the Theorem of Part I.*

From Part I we have, the four reference points and the unit point being taken as base:

Basic R^3's touch the surface

$$(\alpha x)^m = 0$$

at points of the curve of order $m(m+1)$ cut out of $(\alpha x)^m$ by the surface

$$(\alpha x)^{m-1}(\alpha x^2) = 0, \tag{12}$$

and osculate $(\alpha x)^m$ in the $m(m+1)(m+2)$ points of intersection of the three surfaces

$$\left.\begin{array}{c}(\alpha x)^m = 0,\\ (\alpha x)^{m-1}(\alpha x^2) = 0,\\ (m-1)(\alpha x)^{m-2}(\alpha x^2)^2 + 2(\alpha x)^{m-1}(\alpha x^2) = 0.\end{array}\right\} \tag{13}$$

(12) is on all nodes of $(\alpha x)^m$. Hence:

The locus of points of contact of basic R^3's with an m-ic surface passes twice through all nodes of the m-ic. In particular, if the m-ic have a double curve, this curve factors twice out of the curve of contact. A cuspidal curve of the m-ic factors three times out of the curve of contact.

The following is easily verified:

If $(\alpha x)^m$ passes through a base-point, (12) touches $(\alpha x)^m$ there, and the curve of contact has a node at the base-point.

§ 10. *The Basic R^3 Bisecant to a Given Line.*

The known theorem that there is one and only one basic R^3 which meets a given line π of space twice, is an immediate consequence of theorem b). If we choose any plane $\alpha+\lambda\beta$ of the pencil of planes containing π, the R^3 on the base-points and the pole of π as to the conic $C_{\alpha+\lambda\beta}$ in this plane must meet π twice, since the set of points $\iota_3^{(3)}$ determined by this pole on $\alpha+\lambda\beta$ is apolar to $C_{\alpha+\lambda\beta}$. The locus of this pole, as $\alpha+\lambda\beta$ turns around π, may be shown independently to be a basic R^3; we shall do this later. We shall call this basic R^3 the *bisecant* basic R^3 to π.

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[*] This cubic may be regarded as the locus of the lines of the complete 4-point of a set of one of the involutions of quartics on C_α.

[†] Salmon, "Conic Sections," p. 341, Ex. 2.

§ 11. *Cubic Surface Determined by a Line.*

As the plane $\alpha+\lambda\beta$ turns around π the conic $C_{\alpha+\lambda\beta}$ generates a cubic surface on the five base-points, and containing π. If a plane is on a base-point, the configuration B on that plane is merely a complete 4-line taken with the six lines projecting its six points from the base-point, and the conic C_α is in this case the lines ξ, ξ' through the base-point which are partners in the Cremona line-involution determined by the range of conics on the four lines. It follows that the planes joining π to each of the base-points are triple tangent planes of the cubic surface. The degenerate conics $C_{\alpha+\lambda\beta}$ on these planes form the ten lines of the surface that meet π. The conic C_α on a plane α is cut out of the plane by the quadric

$$(\alpha x^2)=0.^*$$

The conic $C_{\alpha+\lambda\beta}$ on a plane of the pencil

$$(\alpha x)+\lambda(\beta x)=0 \tag{14}$$

is cut out of the plane by the corresponding quadric

$$(\alpha x^2)+\lambda(\beta x^2)=0. \tag{15}$$

Eliminating λ from (14) and (15) we have as the locus of $C_{\alpha+\lambda\beta}$ the cubic surface

$$\begin{vmatrix} (\alpha x^2), & (\beta x^2) \\ (\alpha x), & (\beta x) \end{vmatrix} \equiv \Sigma\, \pi_{ij}\, x_i x_j (x_i - x_j) = 0, \tag{16}$$

the cubic surface determined by the line

$$\pi_{ij}=\alpha_i\beta_j-\alpha_j\beta_i$$

and the base.

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* Humbert (*loc. cit.*) gives this form of C_α. It may also be inferred from Reye, "Geometrie der Lage," II, p. 226.

The combinations $x_i x_j (x_i - x_j)$ in (16) are significant. The basic R^3 on x is

$$y_i = \frac{x_i}{1 - x_i t},$$

whence

$$\frac{dy_i}{dt} = \frac{x_i^2}{(1 - x_i t)^2},$$

$t = 0$ giving the point x. It follows that the basic R^3 on x passes through x with the tangent

$$p_{ij} = x_i x_j (x_i - x_j). \tag{17}$$

(16), then, appears as the condition that the tangent at x to the basic R^3 on x shall meet the line π. Hence:

The cubic surface (16) *is the locus of points x such that the tangent at x to the basic R^3 on x meets a given line π.*

If π is a tangent to a basic R^3, the cubic surface (16) has a node at the point of tangency, and the six lines through the node are the line π and the lines to the five base-points. Since an R^3 is projected from any point on it by a quadric cone, we have Cremona's theorem:

The six lines through the node of a nodal cubic surface are generators of a quadric cone.

§ 12. *Degeneration of Basic R^3's.*

It is of importance to point out the way in which basic R^3's may degenerate. Any base-line 12 taken with a conic on 3, 4, 5, and the point where 12 meets 345, is a degenerate basic R^3. A basic R^3 cannot meet 345, except at the points 3, 4, 5, without degenerating in this way. The section of the cubic surface (16) by 345 must, then, be the cubic curve of contact of conics of this pencil with lines of the pencil on the point where π meets 345.

§ 13. *A Ruled Cubic, and Its Connection with* (16).

If a point x of a basic R^3 run along a line p, the tangent at x to the variable R^3 generates a ruled cubic surface containing the line p as directrix.

For the ruled surface thus generated is met by any plane α on the line in the line itself and in the two tangents to basic R^3's at the points where the line meets the conic C_α. We indicate this ruled cubic surface determined by a line p by the symbol $[p]$.

The cubic surface (16) is general and contains twenty-seven lines. We have mentioned eleven of them, the line π and the ten lines meeting it.

Consider any one of the other sixteen, p say. All tangents to basic R^3's at points x of p meet π. π must therefore be contained in the ruled cubic surface $[p]$. Consider a section of $[p]$ by any plane α on π. The line π and the tangent to the basic R^3 on the point where p meets α are the only lines that can be included in this section. Hence the line π must count doubly. Again, a ruled cubic surface cannot contain more than two lines meeting all generators, and one of these must be the double line. If three skew lines meet all generators of a ruled surface the surface is a quadric. We have, therefore, the theorem:

The line π determining the cubic surface (16) *is the double line of any ruled cubic surface, $[p]$, determined by any line, p, of the surface skew to π.*

§ 14. *Complex of Tangents to Basic R^3's.*

We see from (17) that the tangents to all basic R^3's depend on the variation of four homogeneous parameters; that is, they lie in a complex. We proceed to determine its order. Suppose the p_{ij} to be the homogeneous coördinates of a point in five dimensions. The lines of space are represented by points of the quadric spread

$$Q \equiv p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0.$$

(17), considered as a locus of points in five dimensions, is evidently a 3-way spread contained entirely in Q. The four planes x_i and the six $x_i - x_j$ are the ten planes joining the base-points. p_{ij} is the product of three planes on an edge of the tetrahedron of reference. All the p's, considered as cubic surfaces, contain the five base-points and the ten points in which a base-line meets the opposite plane. Hence

$$\Sigma\, a_{ij}\, p_{ij} = 0 \tag{18}$$

is a cubic surface on these fifteen points. Now the order of a 3-way spread in five dimensions is the number of points in which it is met by an arbitrary plane, which may be represented analytically by three equations of the form (18). Three of the cubic surfaces (18) meet in $27 - 15 = 12$ variable points. Hence our complex is represented by a 3-way spread of order 12, and the complex must be of order 6. We have then:

The totality of tangents to basic R^3's are in a sextic complex.[*]

(18) is a system of surfaces on fifteen points, but it is a 5-fold system, as we should expect. The fifteen points have the property that any cubic surface on

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[*] Sturm, *loc. cit.*

fourteen of them must be on the fifteenth, a fact easily verified by actually building up the equation of the cubic surface on fourteen of the points. (18) is merely the locus of points of tangency of basic R^3's with lines of the complex of tangents that lie in a given linear complex. (16) is obviously a special case, the linear complex being there the lines that meet a given line π.

The equation of the complex of tangents to basic R^3's may be obtained by elimination of the x's from (17), but it is more easily found indirectly. The form of equation (15) shows that π, the axis of the pencil of planes $\alpha + \lambda \beta$, is cut by $C_{\alpha+\lambda\beta}$ in pairs of an involution. This fact is also contained in theorem b), which shows that points of $C_{\alpha+\lambda\beta}$ on π are apolar to the two points of intersection with π of its bisecant basic R^3; hence the bisecant R^3 meets π in the double points of this involution. If the bisecant basic R^3 touches π, the two double points of the involution on π coincide, and the equation of the complex of tangents to basic R^3's may therefore be obtained by imposing the condition that this involution be parabolic. Represent the pencil of planes on π by $\alpha + \lambda \beta$, and the range of points by $a + \mu b$. The involution of points where the conics $C_{\alpha+\lambda\beta}$ meet π is cut out by

$$(\alpha x^2) + \lambda (\beta x^2) = 0.$$

The double points of this involution are given by the Jacobian of the quadratics in μ,

$$(\alpha x^2) = (\beta x^2) = 0,$$

when $a + \mu b$ is substituted for x. The discriminant of this Jacobian is

$$\begin{aligned}&[(\alpha a^2)(\beta b^2) - (\alpha b^2)(\beta a^2)]^2 \\ &\qquad - 4[(\alpha a^2)(\beta a b) - (\beta a^2)(\alpha a b)]\,[(\alpha a b)(\beta b^2) - (\beta a b)(\alpha b^2)] = 0,\end{aligned}$$

and writing in this

$$\pi_{ij} = \alpha_i \beta_j - \alpha_j \beta_i = p_{kl}, \quad p_{ij} = a_i b_j - a_j b_i,$$

we have, as the equation of the complex,

$$\sum^{6} p_{12}^4 p_{34}^2 - 2 \sum^{12} p_{21}^2 p_{23}^2 p_{41} p_{43} + 2 \sum^{3} p_{12}^2 p_{34}^2 (p_{14} p_{32} + p_{13} p_{42}) = 0. \qquad (19)$$

§ 15. *A Theorem of Sturm and Its Use.*

There being a double infinity of basic R^3's, if a single condition is imposed on a basic R^3 the R^3 will lie on a surface. A theorem of Sturm is important in this connection:[*]

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[*] Sturm, *loc. cit.*

Two loci of basic R^3's can meet only in basic R^3's and in parts of degenerate basic R^3's (in general, base-lines).

We give Sturm's proof of this, since it applies almost without change to higher cases. If x is a point of intersection of the two surfaces, the basic R^3 on x must lie on both surfaces, since this R^3 is uniquely determined by x, and since, each surface being a locus of basic R^3's, there must be a basic R^3 in each surface through the point x.

As an illustration of the practical application of this theorem, we prove that there are five basic R^3's meeting two lines π and π' — a fact which Sturm reached in another way. Basic R^3's meeting a line π lie on a surface whose order is obviously the number of basic R^3's meeting two lines. Now the surface determined by π contains each of the ten base-lines singly; the basic R^3 of which 12 is a part is 12 taken with the conic on 3, 4, 5 and the points where 12 and π meet 345. Assume now that the two surfaces π and π' meet in n basic R^3's. We have, then, that the total curve of intersection is a curve of order n^2, n being the order of each surface. In this intersection are the ten base-lines and n basic R^3's. This gives the equation

$$n^2 = 3n + 10 \quad \text{or} \quad n = 5.$$

§ 16. *The Locus of Basic R^3's Meeting a Line.*

We have, then, that basic R^3's which meet a given line π lie on a quintic surface. The bisecant basic R^3 to π is obviously a double curve of this surface. The surface contains the ten base-lines and has triple points at the five base-points. It may be pointed out in passing that the surface is rational, and may be considered as mapped from a plane by quartic curves on the ten points of intersection of five lines and on an eleventh point K.* The point K maps into the line π. The five lines in the plane map into the base-points, and their ten points of intersection into the ten base-lines. The two tangents from K to the conic on the five lines map into the basic R^3 bisecant to π, points of these two lines which map into the same point of the surface being determined by tangents to the conic.

This quintic surface is met by a general plane in a quintic curve with 3 nodes. It is met by a plane $\alpha + \lambda\beta$ on π in the line π and a quartic curve with a node at the pole, M, of π as to $C_{\alpha+\lambda\beta}$. It is clear from theorem b) that the joins of corresponding points on this quartic (points lying on the same basic R^3) are polar lines of points of π as to $C_{\alpha+\lambda\beta}$ and hence must pass through M. Since an R^3 is projected from any point of it by a quadric cone, we have:

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* Compare Clebsch, *Math. Ann.*, I.

This quartic is the locus of points on $\alpha+\lambda\beta$ from which the five base-points and the point M are projected by six lines of a quadric cone.

It is the section by $\alpha+\lambda\beta$ of the Weddle quartic surface with nodes at the six points $1, 2, 3, 4, 5, M$, the section of a Weddle quartic by a plane through a node. It meets π in the two points of $C_{\alpha+\lambda\beta}$ and in the two points of intersection with π of its bisecant basic R^3, these four points forming harmonic pairs. Any point P of space determines with regard to an R^3 a point P' such that P and P' are apolar to all quadrics on the R^3. PP' is a bisecant line of the R^3 and P and P' are apolar to the points of intersection of the R^3 with PP'. Since π may be considered as *any* bisecant line to the R^3 on $1, 2, 3, 4, 5, M$, we see from the above that the transformation PP' set up by the R^3 on its six nodes leaves a Weddle surface unaltered.*

The six points of tangency of tangents to the quartic in $\alpha+\lambda\beta$ are on $C_{\alpha+\lambda\beta}$. These are the six tangents through M to the complex curve of (19) on $\alpha+\lambda\beta$. The curve of this complex on any plane α is the polar reciprocal of the sextic of theorem e) in the conic C_α. The cone of rays of the complex through any point M is the enveloping cone from M to the Weddle surface determined by the five base-points and M. The points of contact of basic R^3's are points of contact with the surface.

We have made the statement that the locus of the pole of a line π as to the conic $C_{\alpha+\lambda\beta}$ on a plane $\alpha+\lambda\beta$ containing π, is a basic R^3 meeting π twice. The order of the curve may be found by considering its points of intersection with a plane $\alpha+\lambda\beta$. These points are the pole of π as to $C_{\alpha+\lambda\beta}$, and such points of the curve as may fall on π. For a point of the locus to be on π, π must touch a conic $C_{\alpha+\lambda\beta}$. But this happens twice, since these conics meet π in pairs of an involution. Hence the locus is an R^3 meeting π twice. It is a basic R^3, since the pole as to the degenerate conic on the plane joining π to a base-point is the node of this conic—the base-point itself.

§ 17. *Curves Determined on Surfaces by Basic R^3's.*

We have seen that basic R^3's touch an m-ic surface at points of a curve, F, of order $m(m+1)$, cut out of the m-ic surface by an $(m+1)$-ic surface. We ask for the order of the locus, G, of points in which basic R^3's tangent to an m-ic surface meet this surface again. This may be determined by finding in how many points G meets a base-plane. All points of intersection of G with a base-plane are determined by degenerate basic R^3's, since a basic R^3 cannot meet a base-plane except at a base-point without degenerating. Every base-plane meets the m-ic surface in an m-ic curve, and every base-line meets it in m points. There being $m(m+1)$ conics of a pencil that touch an m-ic curve we have m $m(m+1)$-fold points of G on every base-line, and since there are three base-lines on every base-plane, this gives $3m^2(m+1)$ intersections with G. In addition we have on a base-plane the points in which conics of the pencil which are tangent to the m-ic curve meet this curve again. There are $(2m-2)m(m+1)$ of these points. Hence the order of G is

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* Schoute, *Niew Archief*, (2), 4, (1899), p. 97.

$$3m^2(m+1)+(2m-2)m(m+1)=m(m+1)(5m-2).$$

We have:

Basic R^3's that touch a general m-ic surface meet the surface again in points of a curve, G, of order $m(m+1)(5m-2)$. G has $10m$ $m(m+1)$-fold points at the $10m$ points in which the base-lines meet the surface.

The sextic of theorem e) is a special case of this theorem.

G can meet F, or the $(m+1)$-ic surface cutting out F, in only two ways:

1) At a point of contact of a basic R^3 osculating the m-ic surface;

2) At a point of contact of a basic R^3 bitangent to the surface.

It touches F in the first case; merely intersects in the second. The number of points of intersection of the second kind is

$$m(m+1)^2(5m-2)-2m(m+1)(m+2)=m(m-1)(m+1)(5m-6).$$

Hence:

There are $\frac{1}{2}m(m^2-1)(5m-6)$ basic R^3's bitangent to a general m-ic surface.

The order of G may be obtained in another way. The locus of basic R^3's that meet a given line is a quintic surface containing the base-lines singly. We see from what we have said above that the base-lines are $m(m+1)$-fold lines on the locus of basic R^3's tangent to an m-ic surface. These two loci can meet only in basic R^3's and in base-lines. We have, then, if x is the order of the locus of tangent basic R^3's,

$$5x-10m(m+1)=3x \text{ or } x=5m(m+1).$$

Therefore:

Basic R^3's tangent to an m-ic surface lie on a surface of order $5m(m+1)$.

This surface touches the m-ic along F and cuts out the curve G. Hence the order of G is

$$5m^2(m+1)-2m(m+1)=m(m+1)(5m-2).$$

§ 18. *Ruled Surface Determined by a Curve.*

If a point x of a variable basic R^3 run along an m-ic curve, the tangent to this R^3 at x generates a ruled surface. The order of this surface is the number of generators met by an arbitrary line, π. Now the line π determines a cubic surface (16) which meets the m-ic curve in $3m$ points. The tangent at x to the R^3 on a point x of this surface meets π. Hence:

The locus of the tangent at x to the basic R^3 on a point x of an m-ic curve is a ruled surface of order $3m$.

Since the curve of contact of basic R^3's with an m-ic surface is of order $m(m+1)$, it follows that:

The tangents to basic R^3's tangent to an m-ic surface at points of contact with the surface lie on a ruled surface of order $3m(m+1)$.

If the m-ic surface is a quadric, this ruled surface of contact-tangents is of order 18. It meets the quadric in a curve of order 36. Out of this curve the curve F, of order 6, must factor twice, leaving a curve of order 24 to be accounted for. Now a tangent to a tangent R^3 touches the quadric and can meet in no further point without lying entirely on it. It follows that the curve of intersection of the quadric and ruled surface of contact-tangents is the curve F taken twice, and twenty-four lines. This is also obvious from another point of view. There must be twelve lines of each regulus on the quadric in the sextic complex of tangents to basic R^3's.

§ 19. *Degeneration of the Curve F. The Quintic Surface with Five Triple Points.*

If an m-ic surface contain base-lines or basic R^3's, these will appear in the curve F. For instance, a Weddle surface with nodes at the base-points contains the ten base-lines and the R^3 on the six nodes. The curve F is in this case the ten base-lines, the R^3 on the nodes, and a curve of order $4.5-10-3=7$. It is evident from what we have said that this septimic is the curve of tangency of the enveloping cone from the remaining node.

If the m-ic is a quadric on the base-points, a basic R^3 cannot touch the surface without lying entirely on it, as tangency requires seven points common to the R^3 and the quadric. It is easy to show that there is actually a curve F in this case. A basic quadric is

$$\Sigma\, \alpha_{ij}\, x_i\, x_j = 0,$$

where

$$\Sigma\, \alpha_{ij} = 0.$$

F is cut out by the cubic surface

$$\Sigma\, \alpha_{ij}\, x_i\, x_j\, (x_i + x_j) = 0,$$

which neither vanishes nor factors in general. The curve F can be nothing but basic R^3's. The order of F being 6, we have that *there are two basic R^3's on a basic quadric,* a theorem of Reye. Given a pencil of basic quadrics

$$\Sigma\, \alpha_{ij}\, x_i\, x_j + \lambda\, \Sigma\, \beta_{ij}\, x_i\, x_j = 0, \tag{20}$$

with the conditions

$$\Sigma\, \alpha_{ij} = \Sigma\, \beta_{ij} = 0,$$

the basic R^3's on quadrics of this pencil are cut out by

$$\Sigma\, \alpha_{ij}\, x_i\, x_j\, (x_i + x_j) + \lambda\, \Sigma\, \beta_{ij}\, x_i\, x_j\, (x_i + x_j) = 0. \tag{21}$$

Eliminating λ from (20) and (21) we have as the locus of basic R^3's on quadrics of the pencil (20) the quintic surface

$$\begin{vmatrix} \Sigma\, \alpha_{ij}\, x_i\, x_j, & \Sigma\, \beta_{ij}\, x_i\, x_j \\ \Sigma\, \alpha_{ij}\, x_i\, x_j\, (x_i + x_j), & \Sigma\, \beta_{ij}\, x_i\, x_j\, (x_i + x_j) \end{vmatrix} = 0,$$

or

$$\left.\begin{aligned} &\overset{13}{\Sigma}\, (\alpha_{12}\, \beta_{13} - \alpha_{13}\, \beta_{12})\, x_1^2\, x_2\, x_3\, (x_2 - x_3) \\ &\qquad + \overset{6}{\Sigma}\, (\alpha_{12}\, \beta_{34} - \alpha_{34}\, \beta_{12})\, x_1\, x_2\, x_3\, x_4\, (x_1 + x_2 - x_3 - x_4) = 0. \end{aligned}\right\} \tag{22}$$

This is, as we shall see, *the general quintic surface with five triple points.* It must contain the base-lines singly, since it can meet the locus of basic R^3's which meet a line in only five basic R^3's, — the R^3's on the five points where the line meets the surface, — and hence the ten base-lines must be contained in it to make up the total intersection of order 25 of the two surfaces. A basic R^3 cannot meet (22) without lying entirely on it; hence the fifteen intersections of basic R^3's with the surface must all be at base-points. The base-points are therefore triple points of (22). A quadric of the generating pencil meets the surface in the two basic R^3's on the quadric, and in a basic E^4, the base-curve of the pencil. All of these facts might have been inferred from the equation of the surface.

The locus of the sextic curve of contact of basic R^3's with quadrics of the general pencil,

$$(\alpha x)^2 + \lambda (\beta x)^2 = 0,$$

is the quintic surface

$$\begin{vmatrix} (\alpha x)^2 & (\beta x)^2 \\ (\alpha x)(\alpha x^2) & (\beta x)(\beta x^2) \end{vmatrix} = 0. \tag{23}$$

(23) is on the base-points (singly) and contains the E^4 defined by the pencil. It passes through each of the base-points with the same tangent plane as the quadric of the pencil containing that point. A basic R^3 meets (23) in fifteen points; five are at base-points, and the other ten points are the ten points of tangency of quadrics of the pencil with the R^3. That there are just ten such points is evident from the fact that quadrics of the pencil cut out on the R^3 a binary sextic of the form

$$(\alpha t)^6 + \lambda (\beta t)^6 = 0,$$

and the discriminant of the binary sextic is of degree 10.

The locus of basic R^3's meeting a line is a special case of (22); in this case the pencil of quadrics contains the line and its bisecant basic R^3. We may obtain the equation of this surface in another form. A pencil of basic quadrics on the line joining the points a and b may be written:

$$\Sigma \alpha_{ij} x_i x_j = 0,$$

with the conditions

$$\begin{aligned} &\Sigma \alpha_{ij} = 0, \\ &\Sigma \alpha_{ij} a_i a_j = 0, \\ &\Sigma \alpha_{ij} b_i b_j = 0, \\ &\Sigma \alpha_{ij} (a_i b_j + a_j b_i) = 0. \end{aligned}$$

The two basic R^3's on a quadric of the pencil are cut out by

$$\Sigma \alpha_{ij} x_i x_j (x_i + x_j) = 0.$$

Eliminating α_{ij} from these six equations we have the quintic surface in the form of a 6-row determinant:

$$| x_i x_j, \quad x_i x_j (x_i + x_j), \quad 1, \quad a_i a_j, \quad b_i b_j, \quad a_i b_j + a_j b_i | = 0. \tag{24}$$

The argument which we have used to show that the order of the locus of basic R^3's tangent to an m-ic surface is $5m(m+1)$ may be used to show that the order of the locus of basic R^3's meeting an m-ic curve is $5m$. The same argument shows that the order of a locus of basic R^3's is completely determined by the multiplicity of the base-lines in the surface. If the multiplicity of these lines is m, the order of the surface is $5m$. If the m-ic above is on the base-points, the number of points apart from base-points in which it meets the base-planes is reduced by three, the multiplicity of the base-lines in the surface obtained is reduced by three, and the order of the surface is reduced by fifteen. The order of the locus of basic R^3's meeting a basic E^4 is therefore $5 \cdot 4 - 15 = 5$. Basic quadrics on the E^4 meet one of these R^3's in six points, and hence some quadric of the pencil must contain it entirely. An R^3 on a quadric meets an E^4 on the quadric in six points, the six points in which it meets the quadric cutting out the E^4. Hence the theorem:

The surface (22) *may be considered as the locus of basic R^3's meeting a basic E^4.*

The following theorem is also an easy inference from the argument of this section:

If an m-ic curve have p_i-fold points at the base-points i, the locus of basic R^3's meeting the m-ic is a surface of order $5m - 3\Sigma p_i$, *having the line* 12 *as* $(m - p_3 - p_4 - p_5)$*-fold line, etc.*

Whence:

The locus of basic R^3's meeting an R^3 on four of the five base-points is a 4-nodal cubic surface with nodes at four of the base-points and on the remaining base-point.

§ 20. *Plane Sections of the Quintic Surface with Five Triple Points.*

The section of the surface (22) by a plane α is a curve of interest. We approach this curve from another point of view. By theorem f) a basic E^4 meets any plane α in a set of four points orthic to C_α. Every conic of the pencil, Φ say, on these four points is apolar to C_α. These conics are the sections by α of basic quadrics on the basic E^4. There are two sets of three points $\iota_3^{(3)}$ on each conic of the pencil, since there are two basic R^3's on a basic quadric. The locus of sets $\iota_3^{(3)}$ on conics of the pencil Φ is met by one of these conics in the base-points of the pencil and in the six points of the two sets $\iota_3^{(3)}$,—ten points in all. This shows that the locus is a quintic curve, Q say. Q is on the ten points of the configuration B in α. For, consider the conic Φ on 12. This conic being apolar to C_α cuts 345, the polar line of 12 as to C_α, in a pair of points apolar to C_α. But 12 taken with any two points of 345 which are apolar to C_α is a set $\iota_3^{(3)}$. Hence:

Q is on the ten points of the configuration B in α. It meets a line of the configuration in the three configuration points and in two further points apolar to C_α.

It may be pointed out in passing that these two further points form a 2-point $\iota_2^{(3)}$, *as determined by the configuration B set up by the pencil Φ.*

Since the 3-points $\iota_3^{(3)}$ are apolar to C_α, we have the theorem:

The lines joining 3-points $\iota_3^{(3)}$ inscribed in conics of a pencil orthic to C_α touch a curve, K, of class 5.

For K is the polar reciprocal of the quintic Q in the conic C_α. This gives the further theorem:

Lines bisecant to basic R^3's which lie on quadrics of a basic pencil are in a quintic complex.

For there are ∞^3 of these lines; $\infty^1 R^3$'s and ∞^2 lines bisecant to an R^3. The lines enumerated in this way are all distinct, since there is only one basic R^3 bisecant to a given line. Hence they lie in a complex, and the fact that K is the complex-curve on the plane α gives us the above theorem.

For Q to have a node, in general, would require the surface (22) to have a double curve, which is not true. Hence:

The curves Q and K are quintics of genus 6.

If the pencil Φ is a line m taken with the pencil of lines on its pole, M, as to C_α, the quintic Q reduces to m taken with a quartic with node at M. This quartic, we have seen, is the section by α of the Weddle surface with nodes at 1, 2, 3, 4, 5, M. Similarly the curve K reduces to the point M and a line-quartic with m as a double line.

Since K is the polar reciprocal of Q as to C_α, we have:

K touches the polar lines of the base-points of Φ. It also touches the lines of the configuration B.

Calling the four base-points of Φ a, b, c, d, a degenerate conic $ab - cd$ must contain two 3-points $\iota_3^{(3)}$. Each line ab and cd must contain two points of a set, since every line has a unique bisecant basic R^3. Hence:

K touches the six lines joining the base-points of Φ.

It follows that:

Q passes through the six vertices of the 4-line which is the polar reciprocal as to C_α of the base 4-point of Φ.

We have mentioned the duality of figures associated with a configuration B. The 3-point $\iota_3^{(3)}$ and the 3-line on these points are dual figures. These sets of three lines are determined by an orthic 4-line of B in exactly the dual of the way in which sets $\iota_3^{(3)}$ are determined by an orthic 4-point. There are two points determined on every line of the plane, — the two points of intersection of its bisecant basic R^3. Dually, two lines are determined on every point of the plane,—the two lines to the other points of the set $\iota_3^{(3)}$ determined by that point. These sets of points and lines are determined completely, and in dual ways, by the configuration B, without reference to the base in three dimensions. The two lines through a point a on α may be regarded as the two generators cut by α out of the quadric cone projecting 1, 2, 3, 4, 5 from a.

The relation between Q and K is, of course, completely dual. Q may be regarded as the locus of point-sets $\iota_3^{(3)}$ inscribed in conics of a pencil orthic to C_α, or as the locus of vertices of line-sets circumscribed about conics of a range orthic to C_α. Similarly K may be regarded as the locus of line-sets inscribed in conics of a pencil orthic to C_α, or as the locus of point-sets circumscribed about conics of an orthic range. The pencil and range are polar reciprocals. It is known that two 3-lines inscribed in a conic touch a conic, and dually. A conic of the pencil determines in this way a conic of the range, and *vice versa.* Incidentally we have the theorem:

If a conic (in lines) is apolar to C_α, there are just two line-sets $\iota_3^{(3)}$ circumscribing it. The conic on the vertices of these two 3-lines is apolar (in points) to C_α.

From what we have said we have:

The curve Q meets tangents to the curve K in pairs of points apolar to C_α, and dually.

The following theorem is also evident:

The bisecant basic R^3 meets any line on α in the Jacobian pair of the points in which the line meets C_α and the points in which it is touched by conics of an orthic pencil of B. For the points of the bisecant R^3 are cut out by a conic of an orthic pencil.

The base-points of Φ, a 4-point orthic to C_α, determine a second configuration, B', on α. We may state the facts found above in another way:

A second configuration B' with the same polarity C_α as B determines two quintic curves, a point-quintic Q and a line-quintic K, polar reciprocals as to C_α. Q contains the points of both configurations and cuts the lines of each configuration in a further pair of points apolar to C_α,—in fact, sets $\iota_2^{(3)}$ as determined by the other configuration.

It will be noticed that the two configurations B and B' bear exactly symmetrical relations to the quintics Q and K. The quintics Q and Q' determined by considering B or B' as fundamental meet the lines of the two configurations in the same points, sixty points, and must therefore coincide. Similarly B and B' play symmetrical rôles with regard to K.

§ 21. *Configurations on the Plane Section.*

Q, being entirely symmetrical with regard to both configurations, may equally well be generated by conics on any other orthic 4-point of the configuration B'. These two orthic 4-points determine two basic E^4's with a point in common, — the common point of the two 4-points. The first E^4 is on the surface (22). But the second must also lie on (22). For the new E^4 determines a quintic surface which meets α in the curve Q, since any two orthic 4-points of B' determine the same quintic curve Q. The two quintic surfaces have this curve in common, and hence must contain all basic R^3's meeting this curve; *i. e.*, they must coincide.

The fact that the new E^4 meets the old E^4 in a point on an arbitrary plane α indicates that there are an infinity of basic E^4's on the surface (22). We may see in another way that this is true. Any two basic R^3's lie on one and only one basic quadric. Choose any two basic R^3's on (22) that are not on a quadric of the generating pencil. These R^3's determine a basic quadric which meets (22) in the two R^3's and in a quartic curve, — *basic* since (22) has triple points at base-points. This quartic is a basic E^4. For an E^4 on a quadric cuts the generators of both systems twice; an R^4 on a quadric cuts the generators of one system three times and of the other once. In fact, the unique quadric on an R^4 is the locus of lines trisecant to the curve. Now any generator of the quadric above meets (22) in five points. It meets the pair of R^3's in three points,—the points in which it meets the cubic surface cutting them out. Hence the residual quartic meets every generator of both systems of the cutting quadric twice and must be an E^4. Hence:

The surface (22) *contains an infinity of basic E^4's.*

Again, any basic quadric on an E^4 of (22) cuts out the E^4 and a sextic with five actual nodes. A sextic in space with five nodes projects from any one of these nodes into a plane quartic with four nodes; *i. e.*, it is projected from any one of its nodes by a pair of quadric cones. The sextic must, then, break up into two R^3's on the five nodes. We see therefore that any quadric on a basic E^4 of (22) cuts out a pair of basic R^3's. There are ∞^2 quadrics on pairs of basic R^3's, and ∞^1 quadrics on each basic E^4. Hence:

The system of basic E^4's on (22) *is singly infinite.*

The following theorems are also obvious from what we have said:

(22) *is the locus of basic R^3's on quadrics of the pencil on any basic E^4 of the surface.*

Two E^4's of the system on (22) *have one and only one point in common, and there are two of these E^4's through every point of* (22).

On the plane α:

There are ∞^1 configurations B inscribed in Q and having the same polarity, C_α.

The locus of the lines of these configurations is the quintic K, the polar reciprocal of Q to C_α.

Any quintic surface with triple points at the five base-points is a locus of basic R^3's, since a basic R^3 meeting this surface in one point not a base-point has sixteen intersections with it and must lie on it. The argument used above applies throughout to such a surface, and we have:

The surface (22) *is the general quintic surface with five triple points.*

Let us apply the facts we have obtained to the special surface (24), the locus of basic R^3's meeting a line p. Basic quadrics meet the nodal curve of (24),—the bisecant R^3 to p,—in one point only apart from base-points. Basic quadrics on pairs of basic R^3's of the surface cut out of the surface a system of nodal quartics. The line p meets a quadric of this sort in the two points of the two R^3's on it, and hence these nodal quartics do not meet p. We have said that the quartic section of (24) by a plane on p is identical with a section of a Weddle surface through a node. The configurations inscribed in the section must be inscribed in the quartic, since the nodal quartic space-curves determining them do not meet p. Hence:

The quartic section of a Weddle surface through a node admits ∞^1 inscribed configurations B all having the same polarity. This polarity is defined by the conic on the six points of tangency of tangents from the node.[*]

§ 22. *Further Loci Determined by Basic R^3's.*

We have seen that basic R^3's osculate an m-ic surface f in its $m(m+1)(m+2)$ points of intersection with an $(m+1)$-ic and an $(m+2)$-ic surface, say f' and f'' respectively. The locus of points of osculation of basic R^3's with surfaces of the net

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[*] This is a degenerate case of a more complicated system of configurations; cf. Morley and Conner, "Plane Sections of a Weddle Surface," *Am. Jour.*, July, 1909.

The surface (22) may be regarded as the section by a space of the spread of bisecants to the elliptic quintic in four dimensions. Most of the facts brought out above are immediately evident from this point of view. Surfaces and curves obtainable in this way will be treated by the author in a future paper, of which an abstract will be found in *Bull. Am. Math. Soc.*, March, 1909.

$$f + \lambda\phi + \mu\psi = 0$$

may be obtained by elimination of λ and μ from this equation and

$$\begin{aligned} f' + \lambda\phi' + \mu\psi' &= 0, \\ f'' + \lambda\phi'' + \mu\psi'' &= 0. \end{aligned}$$

Its equation is

$$\begin{vmatrix} f, & \phi, & \psi \\ f', & \phi', & \psi' \\ f'', & \phi'', & \psi'' \end{vmatrix} = 0, \tag{25}$$

a surface of order $3(m + 1)$. Hence:

Basic R^3's osculate surfaces of a net of m-ics in points of a surface of order $3(m + 1)$.

In particular:

Points of osculation of basic R^3's with planes on a point lie on a sextic surface This surface evidently contains the point singly. This may be seen geometrically or from (25).

A "higher null-system"* in space has three characteristics: α, the number of null-planes on a point; β, the number of null-points on a plane; γ, the number of times both null-point and null-plane are incident with a given line. Basic R^3's determine such a null-system; the null-points of a plane are points of osculation of basic R^3's with the plane; the null-plane of a point is the osculating plane at the point to the basic R^3 on the point. The characteristics of this null-system are:

$$\alpha = 1, \quad \beta = 6, \quad \gamma = 5.$$

The value of γ is evident from the fact that the locus of null-points of a sheaf of planes is a surface of order $\alpha + \gamma$; we have found that this is 6.

A more convenient form than (25) for the locus of points of osculation of basic R^3's with planes of a sheaf may be found. The basic R^3 on x is

$$y_i = \frac{x_i}{1 - x_i t},$$

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* Sturm, "Liniengeometrie," Vol. I, p. 78.

whence

$$\frac{d y_i}{d t} = \frac{x_i^2}{(1 - x_i t)^2},$$

$$\frac{d^2 y_i}{d t^2} = \frac{2 x_i^3}{(1 - x_i t)^3}.$$

The osculating plane at x, $(t = 0)$, is:

$$\left.\begin{aligned} \xi_1 &= x_2 x_3 x_4 (x_2 - x_3)(x_3 - x_4)(x_4 - x_2), \\ \xi_2 &= -x_3 x_4 x_1 (x_3 - x_4)(x_4 - x_1)(x_1 - x_3), \\ \xi_3 &= x_4 x_1 x_2 (x_4 - x_1)(x_1 - x_2)(x_2 - x_4), \\ \xi_4 &= -x_1 x_2 x_3 (x_1 - x_2)(x_2 - x_3)(x_3 - x_1). \end{aligned}\right\} \quad (26)$$

The locus of points of osculation of basic R^3's with planes of a sheaf on a point $\varkappa$ is:

$$\varkappa_1 \xi_1 + \varkappa_2 \xi_2 + \varkappa_3 \xi_3 + \varkappa_4 \xi_4 = 0, \qquad (27)$$

where the ξ's have the values given in (26). (27) has triple points at the base-points and contains the ten base-lines singly. It also contains the fifteen lines like

$$x_1 = x_2 - x_3 = 0.$$

These are the fifteen lines of intersection of the ten base-planes besides the ten base-lines. They are parts of degenerate conics in pencils of conics which form parts of degenerate basic R^3's, and it is geometrically evident that they must lie on (27).

Since some plane on a tangent to an R^3 is the osculating plane at the point of tangency, we have:

(27) *contains the septimic curve of contact of tangents from* $\varkappa$ *to the Weddle surface with nodes at the base-points and* $\varkappa$.

Also:

(27) *touches this Weddle surface along this curve of contact.*

For it meets the Weddle surface in a curve of order 24. Out of this curve the ten base-lines factor. The remaining 14-ic curve must be the 7-ic taken twice, since the Weddle surface is the locus of points of intersection with lines on $\varkappa$ of their bisecant basic R^3's, and no bisecant to an R^3 except a tangent can carry an osculating plane.

In a higher null-system with characteristics α, β, γ, the locus of null-points on planes of a pencil is a curve of order $\beta + \gamma$ meeting the axis of the pencil γ times. Hence:

Basic R^3's osculate planes of a pencil in points of a curve of order 11 *meeting the axis, π, of the pencil five times.*

This 11-ic is the partial intersection of two sextic surfaces (27) determined by two points of π. These surfaces meet besides in the twenty-five lines of intersection of base-planes. The 11-ic lies on the cubic surface (16) determined by π. Any cubic surface determined by a line π on x meets (27) in the 11-ic determined by π and in the 7-ic curve of contact of tangents from x to its Weddle surface.

§ 23. *A Quartic Complex Determined by a Quadric.*

If we require that the points of intersection of its bisecant basic R^3 with a line $p_{ij} = \pi_{kl}$ be apolar to a quadric surface

$$(\gamma x)^2 = 0,$$

the line will lie in a complex. Represent as before the pencil of planes on the line by $\alpha + \lambda \beta$, and the range of points by $a + \mu b$. The points of intersection with p of its bisecant basic R^3 are given by the quadratic in μ

$$\Sigma\, a_1 a_2 p_{12} \pi_{12} + \Sigma\, (a_1 b_2 + a_2 b_1)\, p_{12} \pi_{12} \mu + \Sigma\, b_1 b_2 p_{12} \pi_{12} \mu^2 = 0.$$

The points of intersection with γ are given by

$$(\gamma a)^2 + 2 (\gamma a)(\gamma b) \mu + (\gamma b)^2 \mu^2 = 0.$$

The apolarity-condition of these two quadratics is:

$$(\gamma b)^2 \Sigma\, a_1 a_2 p_{12} \pi_{12} - (\gamma a)(\gamma b) \Sigma\, (a_1 b_2 + a_2 b_1)\, p_{12} \pi_{12} + (\gamma a)^2 \Sigma\, b_1 b_2 p_{12} \pi_{12} = 0,$$

or, by writing $a_i b_j - a_j b_i = p_{ij}$,

$$\begin{aligned} \overset{6}{\Sigma} \, [\gamma_{33} p_{13} p_{23} + \gamma_{44} p_{14} p_{24} - p_{12} (\gamma_{12} p_{12} + \gamma_{13} p_{13} + \gamma_{14} p_{14} \\ + \gamma_{23} p_{23} + \gamma_{24} p_{42}) + \gamma_{34} (p_{13} p_{24} + p_{14} p_{23})]\, p_{12} \pi_{12} = 0. \qquad (28) \end{aligned}$$

Hence:

Lines met by their bisecant basic R^3's in a pair of points apolar to a given quadric, lie in a quartic complex.

A special case of this complex is the complex of lines bisecant to basic R^4's and lying in a given space. We shall recur to this in Part IV of this paper.

IV.

BASIC NORM-CURVES IN FOUR DIMENSIONS.

§ 24. *Introductory.*

The base in a space of four dimensions is a set of six points. Given the set of six points, we have the set of ∞^3 rational quartics on them. We shall call quadric spreads of the linear 8-fold system on the six points *basic* quadric spreads. The theorems a), b), c), of Part III, may be modified for the 4-dimensional case as follows:

a′) *Basic quadric spreads meet any space, α, in quadric surfaces apolar to a definite quadric, Q_α, in α.**

b′) *Basic R^4's meet α in 4-points, $\iota_4^{(4)}$, apolar to Q_α.*

c′) *Basic R^4's tangent to α touch at points of Q_α.*

The quadric Q_α is the quadric with regard to which the $(15_6, 20_3)$ configuration, Γ say, cut out of the complete 6-point by α, is self-polar. As in Part III we name the points of the base 1, 2, 3, 4, 5, 6: we have then fifteen base-lines, 12, etc.; twenty base-planes, 123, etc.; fifteen base-spaces, 1234, etc. We name the points, planes, and lines of Γ after the lines, spaces, and planes determining them.

As before, an immediate consequence of theorem b′) is:

There is one and only one basic R^4 which meets a given plane, π, three times.

This R^4 is the locus of the pole of π as to the quadrics $Q_{\alpha+\lambda\beta}$ on spaces of the pencil $\alpha+\lambda\beta$ containing π. We shall call it the basic R^4 *trisecant* to π.

We shall make some use of degenerate basic R^4's, and it will be well to point out how they may degenerate. Any base-line 12, taken with an R^3 on 3, 4, 5, 6, and 12/3456, the point where 12 meets 3456, is a degenerate basic R^4. There are also pencils of conics that may be regarded as parts of degenerate basic R^4's. The two planes 123 and 456 have a point 123/456 in common. Any conic on 1, 2, 3, 123/456, taken with a conic on 4, 5, 6, 123/456, is a degenerate basic R^4.

We shall say that an R^3 is *orthic* to a quadric Q when every quadric on the R^3 is apolar to Q, or, which amounts to the same thing, when there is *one* 4-point, and hence ∞^1, inscribed in the R^3 and apolar to Q.† Given an R^3

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* Loria, *loc. cit.*

† Reye: "Geometrie der Lage," Part II, p. 226.

orthic to Q, one of these 4-points may be constructed by choosing any point, a, of the R^3 and taking with a the three points of intersection with the R^3 of the polar plane of a as to Q. It is three conditions on an R^3 to be orthic to a quadric, since it is three conditions on the 3-point in the polar plane of a to be apolar to the section of this plane with Q.

§ 25. *The Point-Sets* $\iota_4^{(4)}$.

Project the system of basic R^4's from a base-point, say 6. This gives a system of cubic cones (2-ways). In each of these cones there is a single infinity of basic R^4's, one through every point of the cone. For a basic R^4 cannot meet one of these cones in a point, a, without lying entirely on it, since projection from 1, 2, 3, 4, 5, or a would give an R^3 with seven intersections with a quadric cone. It follows:

There are ∞^2 *such cubic cones projecting basic* R^4*'s from a base-point,* 6. *These cones meet* α *in* R^3*'s of a basic system on* 16, 26, 36, 46, 56.

If every quadric on five points is apolar to a quadric, Q, we may call the set of five points *orthic* to Q. A defining characteristic of such a set of five points is that the polar line as to Q of the line joining any two of the points is on the plane of the opposite three. Now the polar line of 16-26, or 126, as to Q_α, by the well-known properties of Γ, is 345, which is on 36-46-56, or 3456. 16, 26, 36, 46, 56 are, therefore, an orthic 5-point as to Q_α, and every quadric on them is apolar to Q_α. It follows that every R^3 on them is orthic to Q_α. We have from the above the theorem:

4-*points,* $\iota_4^{(4)}$, *in* α *are* 4-*points inscribed in* R^3*'s of the basic system on* 16, 26, 36, 46, 56, *and apolar to* Q_α.

An $\iota_4^{(4)}$ may be constructed when one point, m, is given by drawing the R^3 on 16, 26, 36, 46, 56, m, and finding its three intersections with the polar plane of m as to Q_α. Theorem c′) follows. Also:

Basic R^4*'s osculate* α *in points of a sextic curve* F, *the curve of contact of* R^3*'s on* 16, 26, 36, 46, 56 *with* Q_α.

Basic R^4*'s hyperosculate* α (*have* 4-*point contact*) *in points of osculation of these basic* R^3*'s with* Q_α.

We shall call this set of twenty-four points the points H.

R^3*'s on the six orthic bases of* Γ *define the same curve* F, *and the same twenty-four points of osculation.*

For there are six orthic 5-points in Γ, which we may call the sets $1, \ldots, 6$. Sets $\iota_4^{(4)}$ may be considered as defined by any one of the six systems.

If in S_4 we take as vertices of the pentahedron of reference the five points $1, \ldots, 5$, with 6 as the unit point, we have from Part I:

Basic R^4's touch an m-ic spread in points of its $m(m+1)$-ic 2-way intersection with a spread of order $m+1$.

Basic R^4's osculate an m-ic spread in points of a curve of order $m(m+1)(m+2)$, its curve of intersection with an $(m+1)$-ic and an $(m+2)$-ic spread.

Basic R^4's hyperosculate an m-ic spread in its $m(m+1)(m+2)(m+3)$ points of intersection with an $(m+1)$-ic, an $(m+2)$-ic and an $(m+3)$-ic spread.

In particular:

Basic R^4's touch a space, α, in points of the quadric cut out by

$$(\alpha x^2) = 0, \tag{1}$$

this being an analytical expression for the quadric Q_α.

Basic R^4's osculate α in the points of intersection of Q_α with

$$(\alpha x^3) = 0.$$

This is a curve, F, of order 6, — the complete intersection of a quadric and a cubic surface in α.[*]

Basic R^4's hyperosculate α in the twenty-four points, H, cut out of F by

$$(\alpha x^4) = 0.$$

The points H are the twenty-four points of intersection of a quadric, a cubic and a quartic surface in α.

§ 26. *Further Loci in α.*

Several loci in α besides those we have mentioned will be of use to us.

A. We may ask for the locus of the remaining point of intersection with α of basic R^4's osculating α.

B. We may ask for the locus of the points in which basic R^4's bitangent to α touch α. This locus will lie on Q_α.

C. We may ask for the locus of the two further points of intersection with α of basic R^4's tangent to α. C will be a surface; A and B will be twisted curves in α.

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[*] The curve F is special, but of genus 4. For an account of the properties of such a curve see Pascal, "Rep. der höheren Math.," II, p. 276.

The order of A is easily determined by finding its intersections with a plane of the configuration Γ. A basic R^4 cannot meet a base-space, and hence cannot meet a plane of Γ without degenerating. The R^4's degenerating into conics cannot be made to osculate α. Consider the R^4's made up of the base-line 12 and R^3's on 3, 4, 5, 6, 12/3456. There being six of these R^3's that osculate the plane 3456, the point 12 counts as a 6-fold point of A. Six points 12 are on every plane of Γ. Hence:

The curve A is of order 36.

A is of genus 4, since it is in one-one correspondence with F. It cannot meet Q_α except at points where an R^4 hyperosculates, and hence it must osculate Q_α at each of the twenty-four points H.

The order of B may be determined in a similar way. Only those basic R^4's which degenerate into pairs of conics, say on 123 and 456, can be made to touch α twice and meet a plane of Γ. Each point of contact counts in two R^4's, and hence on each line of Γ there are two nodes of B. There being four lines on each plane of Γ, we have:

The curve B is of order 16.

We now show that *B actually breaks down into two octavics.* Considered in α, B is the locus of points of intersection with lines of Q_α of their bisecant basic R^3's. The locus of points of intersection with lines of a regulus of their bisecant basic R^3's is an octavic on the regulus. For, calling the base in α 1, 2, 3, 4, 5, consider where this curve can meet the plane 123. 45 meets the regulus in two points, thus determining two lines on it. To each of the two points corresponds a point on its line in 123. There are besides, by the same argument, two points of the curve on each of 12, 23, 31. Hence the curve meets 123 in eight points, and is an octavic. It meets a tangent plane to the quadric in eight points: two on one generator and six on the other. It projects from a point of the quadric into a plane octavic curve with one 6-fold point and one double point. Hence it has sixteen apparent double points, and its genus is 5.

B is therefore two octavics, the two curves determined by the two sets of generators on Q_α. Consider the two octavics as projected from a point of Q_α. The thirty-two apparent double points of the two curves, together with their sixty-four intersections, real or apparent, make up the necessary number of double points. There are $2.6.2$ apparent intersections along the two generators through the point of Q_α. The other forty intersections are real, two on each line of Γ.

B can meet (αx^2) only where it meets the sextic F, and this can be only in the twenty-four points H. Hence B touches (αx^2) and therefore F at these points.

Each of the two octavics of B is in one-one correspondence with itself, the points of contact of one basic R^4 being considered as corresponding points. Suppose the R^4

$$x_i = \frac{\varkappa_i}{1 - \varkappa_i t}$$

touches α at $t = t_1$ and $t = t_2$. We must then have the following relations:

$$\Sigma \frac{\alpha_i \varkappa_i}{1 - \varkappa_i t_1} = \Sigma \frac{\alpha_i \varkappa_i}{1 - \varkappa_i t_2} = \Sigma \frac{\alpha_i \varkappa_i^2}{(1 - \varkappa_i t_1)^2} = \Sigma \frac{\alpha_i \varkappa_i^2}{(1 - \varkappa_i t_2)^2} = 0. \tag{2}$$

The coördinates of any point on the line joining the two points of contact may be written:

$$y_i = \frac{\varkappa_i}{1 - \varkappa_i t_1} + \lambda \frac{\varkappa_i}{1 - \varkappa_i t_2}.$$

Now

$$\begin{aligned}(\alpha y^2) &= \Sigma \alpha_i \left(\frac{\varkappa_i}{1 - \varkappa_i t_1} + \lambda \frac{\varkappa_i}{1 - \varkappa_i t_2} \right)^2 \\ &= \Sigma \frac{\alpha_i \varkappa_i^2}{(1 - \varkappa_i t_1)^2} + \frac{2\lambda}{t_1 - t_2} \Sigma \left(\frac{\alpha_i \varkappa_i}{1 - \varkappa_i t_1} - \frac{\alpha_i \varkappa_i}{1 - \varkappa_i t_2} \right) + \lambda^2 \Sigma \frac{\alpha_i \varkappa_i^2}{(1 - \varkappa_i t_2)^2}.\end{aligned}$$

This vanishes for all values of λ on account of the conditions (2). Hence:

The lines joining corresponding points of the curve B lie entirely on the quadric Q_α, which is, as we have seen, obvious from geometrical considerations.

Corresponding points of B coincide at points where basic R^4's hyperosculate. Since B touches F at these points, we have:

The points H are points of F such that the tangents to F at these points are generators of Q_α; that is, the twenty-four points of tangency of tangents to F that meet F again.[*]

We find the order of the surface C in a similar way, namely, by finding the order of the curve in which it meets a plane of Γ, say 1234. The R^3's on five points in the space 1234 touch the plane 1234 along a conic, and cut again along a sextic curve which is on C. Further, the four lines 123, etc., count doubly, since any point on the line 123 is on a conic on 1, 2, 3, 123/456, which, taken with either of two conics on 4, 5, 6, 123/456, satisfy our conditions. Hence:

The surface C is of order 14.

It touches Q_α along the sextic F and cuts out the curves B.

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[*] Pascal, *loc. cit.*

§ 27. *Analytical Treatment of A, B, and C.*

The 14-ic C is analogous to the sextic of theorem e), Part III. We may find its equation referred to an orthic base of Γ by a method similar to that used there in finding the equation of the sextic. Consider 16, 26, 36, 46, 56 as base in α. Take as reference points 16, 26, 36, 46, with 56 as unit point. A basic quadric may be written

$$\Sigma a_{ij} x_i x_j = 0, \tag{3}$$

with the condition

$$\Sigma a_{ij} = 0. \tag{4}$$

The quadric Q_α must be of the form

$$\Sigma a_i \xi_i^2 + 2\lambda \Sigma \xi_i \xi_j = 0, \tag{5}$$

the apolarity-condition of (3) and (5) being merely (4) multiplied by λ. (5) in points is

$$\begin{aligned} Q \equiv \Sigma\, [2\lambda^3 - \lambda^2(a_2 + a_3 + a_4) + a_2 a_3 a_4]\, x_1^2 \\ - 2\Sigma\lambda(\lambda - a_3)(\lambda - a_4)\, x_1 x_2 = 0. \end{aligned} \tag{6}$$

If we find the condition that the polar plane of a point x as to Q touch the basic R^3 on x, we will evidently have the equation of C. The R^3 on x is, parametrically,

$$y_i = \frac{x_i}{1 - x_i t}.$$

This meets the polar plane of x as to Q in the points given by the roots of the cubic in t

$$\left.\begin{aligned} x_1 x_2 x_3 x_4 \Sigma \frac{\partial Q}{\partial x_1} t^3 - \Sigma x_1 x_2 x_3 \left(\frac{\partial Q}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial Q}{\partial x_3}\right) t^2 \\ + \Sigma x_1 x_2 \left(\frac{\partial Q}{\partial x_1} + \frac{\partial Q}{\partial x_2}\right) t - 2Q = 0. \end{aligned}\right\} \tag{7}$$

(7) may be written

$$x_1 x_2 x_3 x_4 L t^3 - f_4 t^2 + f_3 t - 2Q = 0, \tag{8}$$

where L is the polar plane of 56 as to Q, and f_3 and f_4 are a cubic and a quartic surface respectively. (8) gives the parameters along the R^3 on x of the other three points of the set $\iota_4^{(4)}$ determined by x. The parameter of x is 0. Hence we have:

Basic R^4's touch α where $Q = 0$.
Basic R^4's osculate α where $Q = f_3 = 0$.
Basic R^4's hyperosculate α where $Q = f_3 = f_4 = 0$.

In other words, f_3 is a cubic surface on the curve F, and f_4 is a quartic surface cutting out of F the points H.

C is the discriminant of (8):

$$C \equiv 108\, x_1^2 x_2^2 x_3^2 x_4^2 L^2 Q^2 + 4\, x_1 x_2 x_3 x_4 L f_3^3 - 36\, x_1 x_2 x_3 x_4 L Q f_3 f_4 \\ + 8 f_4^3 Q - f_3^2 f_4^2 = 0. \qquad (9)$$

C is of the form

$$\Phi Q + \psi f_3^2 = 0,$$

and hence touches Q along its intersection with f_3. The further intersection with Q is the curve B. B is therefore cut out of Q by

$$\Psi = 4\, x_1 x_2 x_3 x_4 L f_3 - f_4^2 = 0.$$

This form shows that B touches F at the points H; in fact Ψ touches f_3 all along the curve cut out by f_4. The form of Ψ shows that B has nodes at its intersections with the planes of Γ.

The curve A must lie on C. We can find from (8) a surface that will cut it out. If x is a point of A, (8) must have three equal roots, and its Hessian must vanish identically. This Hessian is

$$\phi_8 t^2 + \phi_7 t + \phi_6 = 0,$$

where

$$\phi_8 = 3\, x_1 x_2 x_3 x_4 L f_3 - f_4^2,$$
$$\phi_7 = 18\, x_1 x_2 x_3 x_4 L Q - f_3 f_4,$$
$$\phi_6 = 6 f_4 Q - f_3^2.$$

ϕ_6 is the locus of point-sets $\iota_4^{(4)}$ which are a self-apolar quartic on the basic R^4 (or basic R^3) determining them.* The g_3 of this quartic is

$$9 f_3 f_4 Q - 54\, x_1 x_2 x_3 x_4 L Q^2 - f_3^3 = 0.$$

This is the locus of sets $\iota_4^{(4)}$ that are harmonic pairs on the R^4 determining them. It osculates Q along the sextic F, as is evident geometrically. The above equation has the form

$$f_3 \phi_6 - 3 Q \phi_7 = 0.$$

Hence g_3 contains the curves common to ϕ_6 and ϕ_7.

A is the curve common to ϕ_6, ϕ_7, and ϕ_8. ϕ_6 and ϕ_7 meet in A and the curve F. ϕ_6 has nodes at the points of the orthic 5-plane $x_1 x_2 x_3 x_4 x_5 L$, and ϕ_7 has triple points at these points. A, therefore, has 6-fold points at the points of Γ. ϕ_6 and ϕ_7 meet in A and the curve common to f_3 and f_4. C is obviously, to within a factor,

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* See Part V, § 36.

$$4\phi_6\phi_8 - \phi_7^2 = 0.$$

ϕ_6, then, touches C along the curves F and A. Indeed, the form of ϕ_8 shows that it touches Q, and hence C, along the curve F.

C may also be put into the form

$$x_1 x_2 x_3 x_4 L \Phi + \Psi f_4^2 = 0.$$

This shows that f_4 meets C at its points of intersection with $x_1 x_2 x_3 x_4 L$. These five planes form an orthic 5-plane of Γ. It may be easily verified that f_4 has nodes at the ten points of this 5-plane. Hence:

The points H are on a 10-nodal quartic surface containing the ten lines of any orthic 5-plane of Γ and with nodes at the points of this 5-plane. There are six such quartic surfaces.

The surface f_3 may easily be shown to be on the ten points of an orthic 5-plane. Hence:

The curve F is on a cubic surface with the ten points of any orthic 5-plane of Γ.

§ 28. *A Cubic Spread Determined by a Plane in S_4.*

We see from (1) that quadrics $Q_{\alpha+\lambda\beta}$ on spaces of the pencil,

$$(\alpha x) + \lambda(\beta x) = 0,$$

are cut out by the quadric spreads

$$(\alpha x^2) + \lambda(\beta x^2) = 0.$$

Eliminating λ between these two equations we have, as the locus of quadrics $Q_{\alpha+\lambda\beta}$,

$$\begin{vmatrix} (\alpha x^2), & (\beta x^2) \\ (\alpha x), & (\beta x) \end{vmatrix} \equiv \Sigma\, \pi_{ij}\, x_i x_j (x_i - x_j) = 0,^* \qquad (10)$$

where $\pi_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$ are the coördinates of the plane common to the spaces of the pencil. It is easy to show that the basic R^4 on x passes through x with the tangent

$$p_{ij} = x_i x_j (x_i - x_j).$$

Hence we have:

The locus of points of tangency with basic R^4's of tangent lines meeting a plane π is the cubic spread (10).

---

* This seems to be the *general* cubic spread in S_4 which contains a plane.

In what follows we shall denote this spread by $[\pi]$. It contains the plane π singly.

Consider two cubic spreads $[\pi]$ and $[\pi']$, where the planes π and π' have a line, p, in common. They intersect in the quadric Q_α on the space α containing π and π', and in a septimic 2-way $[p]$, the locus of points of tangency with basic R^4's of tangents meeting the line p. Hence:

The locus of points of tangency with basic R^4's of tangent lines that meet a given line, p, is a 7-ic 2-way, $[p]$.

$[p]$ is met by a space α on p in p and a sextic curve on Q_α. This sextic curve is easily defined by the basic R^3's in α. It is the intersection with Q_α of the cubic surface (16), Part III, determined by the line p. It follows that the sextic curve meets p twice.

Consider now a line, p, and a plane, π, meeting it in a point, a. The spreads $[p]$ and $[\pi]$ meet in a 21-ic curve. Out of this curve the above sextic on the space πp factors. The residual 15-ic curve is the locus of points of tangency of basic R^4's with tangents passing through a. Hence:

The locus of points of tangency with basic R^4's of tangents passing through a point, a, is a 15-ic curve $[a]$.

$[\pi]$ contains π, $[p]$ contains p, and $[a]$ contains a, each singly. Each locus is basic.

§ 29. *The Congruence of Tangents to Basic R^4's.*

There are ∞^1 lines tangent to a basic R^4, and ∞^3 basic R_4's; ∞^4 lines in all. Hence these lines are in a congruence. The order of $[a]$ shows that the order of this congruence is 14, since the congruence cone on any point a is the cone projecting $[a]$ from a. Congruence lines in a space α are tangents to basic R^3's in α at points of a quadric Q_α. Congruence lines meeting a line p are tangents to basic R^3's at points of a sextic curve on Q_α. By a theorem of Part III, these lines lie on a ruled surface of order $6 \cdot 3 = 18$. Now the lines of an (m, n) congruence that meet a given line, p, lie on a ruled surface of order $m + n$. The class of our congruence is therefore 4. We have:

There are four basic R^4's that touch a given plane, π.

Lines tangent to basic R^4's and in a space α are in a (14, 4) *congruence.*

We next determine its *rank*. It is known that the lines of an (m, n, r) congruence that meet a line p lie on a ruled surface of order $m + n$, with p as an m-fold line, and that this surface has an additional double curve of order

$\frac{n(n-1)}{2}+r$. Cut the 18-ic surface, on a space α and determined by the lines of our congruence meeting a line, p, by a plane π on α. This gives an 18-ic curve on π which is in one-one correspondence with the sextic curve of contact of basic R^4's with the lines of our surface, and hence is of genus 4. This 18-ic has a 14-fold point and besides

$$\tfrac{1}{2}\,.\,17\,.\,16-\tfrac{1}{2}\,.\,14\,.\,13-4=41$$

nodes. These are the points in which π meets the additional double curve of the surface, this curve being of order $6+r$. Hence:

The rank of the congruence of lines tangent to basic R^4's and on any space α, is 35.

From what we have said we have the further theorem:

The locus of tangents at points of a line p to basic R^4's meeting p is a cubic 2-way having p for a directrix.

For the line p meets a cubic spread $[\pi]$ in three points, and these points determine the points in which π meets our 2-way. Also:

The locus of tangents at points of a plane π to basic R^4's meeting π is a ruled spread of order 7.

For π meets a 7-ic 2-way $[p]$ in seven points, and the tangents to basic R^4's at these points determine the points in which p meets our spread.

§ 30. *Loci Determined by Systems of Spreads.*

The points of contact of basic R^4's with spreads of the pencil of m-ics,

$$f+\lambda\phi=0,$$

are cut out by $(m+1)$-ic spreads

$$f'+\lambda\phi'=0.$$

Eliminating λ, we have the $(2m+1)$-ic spread

$$\begin{vmatrix} f, & \phi \\ f', & \phi' \end{vmatrix}=0.$$

Hence:

Points of contact of basic R^4's with spreads of a pencil of m-ics are on a spread of order $2m+1$.

Similarly:

Points of osculation of basic R^4's with spreads of a net of m-ics,

$$f+\lambda\phi+\mu\psi=0,$$

are on a spread of order $3m+3$, *of the form*

$$\begin{vmatrix} f, & \phi, & \psi \\ f', & \phi', & \psi' \\ f'', & \phi'', & \psi'' \end{vmatrix} = 0. \tag{11}$$

Let us apply this to a particular case. If f is a basic quadric spread, the curve of osculation of basic R^4's with f, Φ say, is a curve of order 24. But a basic R^4, having already six points in common with f, cannot osculate without lying entirely on f. Φ must therefore be six basic R^4's. We have, then:

There are six basic R^4's on a basic quadric spread.

We should expect, by counting of constants, to find a finite number of basic R^4's on a basic quadric spread. For it is nine conditions on an R^4 to lie on a quadric spread, and the spread therefore contains $\infty^{21-9} = \infty^{12}$ R^4's. These twelve remaining degrees of freedom are taken up by requiring the R^4's to be on six specified points of the spread, since it is two conditions on a curve in a 3-dimensional space (linear or not) to be on a point. This gives us at once:

If f, ϕ, and ψ are basic quadric spreads, (11) *is the locus of basic R^4's lying on spreads of the net* $f+\lambda\phi+\mu\psi=0$. (11) *is in this case of order* 9.

Let us further specialize by making $f+\lambda\phi+\mu\psi=0$ the net of quadric spreads on a plane π and the six base-points. All spreads of the net must contain the basic R^4 trisecant to π, since this R^4 meets any of these spreads in $6+3=9$ points. Now a quadric spread in S_4 containing a plane must have a node and contain two systems of planes on the node, and an R^4 on it meets all planes of one system three times and of the other system once. All R^4's on the spread (11), then, meet π. Again, a basic R^4 meeting π must be contained in some quadric spread of the net, namely that particular quadric spread containing any two further points of it. Hence:

The locus of basic R^4's meeting a plane π is a spread of order 9. *It has the trisecant basic R^4 to π for a triple curve. Its equation is of the form* (11).

It follows that:

There are nine basic R^4's meeting a plane and a line.

Basic R^4's meeting a line lie on a 2-way of order 9.

§ 31. *Some Enumerative Results.*

There being ∞^3 basic R^4's, there are in general a finite number satisfying a given 3-fold condition. We can, by means of the knowledge we have of the curves A and B and the surface C, taken with the theorems just enunciated, find the number of basic R^4's satisfying a variety of 3-fold conditions. We introduce the following symbolism. Denote by:

P_6 the 18-fold condition that an R^4 be basic,
P the 3-fold condition that an R^4 be on a seventh point,
ν the 2-fold condition that an R^4 meet a given line,
μ the 1-fold condition that an R^4 meet a given plane,
ρ the 3-fold condition that an R^4 touch a given plane,
α_2 the 1-fold condition that an R^4 touch a given space,
α_3 the 2-fold condition that an R^4 osculate a given space,
α_4 the 3-fold condition that an R^4 hyperosculate a given space.

A basic R^4 subjected to a λ-fold condition τ has for locus a $(4-\lambda)$-way spread which we shall denote by $[\tau]$. By the theorem of Sturm enunciated in Part III, two spreads $[\tau]$ and $[\tau']$ can intersect only in basic R^4's and in parts of basic R^4's — base-planes, lines, or conics. In order to find the order of the spread of intersection of two of these spreads apart from base-lines, conics on base-planes, and base-planes, it is necessary to know the multiplicity of the base-lines, conics and planes in these spreads. The multiplicity of the base-points in a given 3-way is immediate from the fact that basic R^4's cannot meet the spread except at base-points without lying entirely on it. The following table gives the multiplicities of base-lines, planes and conics in a number of these spreads. Accented letters are meant to indicate that the symbol repeated refers to a different element; indices, that the same element is repeated.

Spread.	Dimension.	Mul. of 15 base-lines.	Mul. of 20 base-planes.	Base-Conics.	Order.
$[\nu]$	2	1	0	0	9
$[\mu]$	3	3	1	0	9
$[\alpha_2]$	3	6	2	0	18
$[\alpha_3]$	2	6	0	0	54
$[\alpha_2^2]$	2	0	0	2 double conics on each plane	32
$[\mu^2]$	2	1	0	1 conic on each plane	17
$[\mu\,\mu']$	2	5	0	2 conics on each plane	61
$[\alpha_2\,\mu]$	2	10	0	1 double conic and 2 single conics on each plane	122
$[\alpha_2\,\alpha_2']$	2	20	0	4 double conics on each plane	244

We now verify the facts contained in this table.

We have shown the order and dimension of $[\nu]$ to be 9 and 2 respectively. It contains the base-lines singly, since the unique R^3 on say 1, 2, 3, 4, 56/1234, and the point where the line ν meets 1234, taken with 56, lies in the spread.

We have seen also that $[\mu]$ is a 3-way of order 9. The plane μ meets a base-plane 123 a point, and the conic of the pencil on 123 on this point, taken with any conic of the pencil on 456, is a basic R^4 meeting μ. A pencil of conics covers a plane singly (there is a unique conic on every point of the plane) and hence $[\mu]$ contains the base-lines singly. $[\mu]$ and $[\nu]$ must meet in nine basic R^4's, the nine R^4's meeting μ and ν. They meet in a curve of order 81. This leaves a curve of order $81 - 36 = 45$ to be accounted for by the fifteen base-lines. Hence $[\mu]$ must contain the base-lines triply, since $[\nu]$ contains them singly.

$[\alpha_2]$ is a 3-way of order 18, since $[\nu]$ meets Q_α in eighteen points and the basic R^4's on these points determine eighteen points in which ν meets $[\alpha_2]$. Two conics of the pencil on 123 touch α, and hence every conic of the pencil on 456 counts doubly in this spread. $[\alpha_2]$ and $[\nu]$ meet in eighteen basic R^4's. They meet in a curve of order 162, in which the base-lines form a part of order $162 - 72 = 90$. Hence $[\alpha_2]$ contains the base-lines 6-fold. It follows:

The points of the configuration Γ *in* α *are sextuple points of the surface* C.

$[\alpha_3]$ is a 2-way of order 54, since $[\mu]$ meets the sextic F in α in fifty-four points. Six R^3's of the basic system in 1234 osculate α. Hence the base-lines are 6-fold.

$[\alpha_2^2]$ meets α in B taken twice. Its order is therefore 32. Two conics of each of the pencils 123 and 456 touch α and these must count doubly in $[\alpha_2^2]$.

$[\mu^2]$ contains the base-lines singly, since there is a unique basic R^3 in 1234 bisecant to the line in which μ meets 1234. It contains on each plane the conic on the point in which μ meets the plane. If x is its order, we have, by combination with $[\mu]$,

$$9\,x - 15\,.\,3 - 20\,.\,2 = 4\,x, \text{ whence } x = 17.$$

$[\mu\mu']$ contains the base-lines 5-fold, since five basic R^3's meet two lines. It contains the two conics on the points in which μ and μ' meet a base-plane. If x is its order, we have, by combination with $[\mu]$,

$$9\,x - 15\,.\,3\,.\,5 - 4\,.\,20 = 4\,x, \text{ whence } x = 61.$$

$[\alpha_2 \mu]$ contains the base-lines 10-fold, since ten basic R^3's meet a line and touch a plane.* Two conics on 123 and two on 456 touch α. Either of the tangent conics on 123 taken with the conic on 456 on the point in which μ meets 456 is a basic R^4 satisfying the conditions. The latter counts doubly, the former singly. Combination with $[\mu]$ gives the order.

$[\alpha_2 \alpha_2']$ contains the base-lines 20-fold, since twenty basic R^3's touch two planes.* Two conics on each base-plane touch each space α and α', and hence there are four double conics on each base-plane. Combination with $[\mu]$ gives the order.

The following table gives the number of basic R^4's satisfying various 3-fold conditions. Most of these may be obtained by combination of the appropriate spreads given in the foregoing table, together with the application of the theorem of Sturm. Others are added for the sake of completeness.

$$\begin{array}{llll}
P_6 P = 1, & P_6 \alpha_2 \nu = 18, & P_6 \alpha_2^2 \mu = 32, & \\
P_6 \mu^3 = 1, & P_6 \alpha_3 \mu = 54, & P_6 \alpha_2^2 \alpha_2' = 64, & \\
P_6 \rho = 4, & P_6 \alpha_2 \alpha_3' = 108, & P_6 \mu \mu' \mu'' = 61, & P_6 \alpha_2 \alpha_2' \alpha_2'' = 488. \\
P_6 \alpha_4 = 24, & P_6 \mu^2 \mu' = 17, & P_6 \alpha_2 \mu \mu' = 122, & \\
P_6 \mu \nu = 9, & P_6 \mu^2 \alpha_2 = 34, & P_6 \alpha_2 \alpha_2' \mu = 244, &
\end{array}$$

The fact that some of these results may be obtained in more than one way furnishes an easy check on the accuracy of the first table.

We may add to the above the following extensions of theorems of Part III:

Basic R^4's meeting a 2-way of order m lie on a 3-way of order $9m$.

Basic R^4's meeting a curve of order m lie on a 2-way of order $9m$.

Modifications of these theorems will be necessary if the director spread is related to the base.

§ 32. *Loci Determined on a Plane by Basic R^4's.*

Consider a pencil of spaces

$$(\alpha x) + \lambda (\beta x) = 0,$$

having a plane, S, as axis. The quadric spreads cutting out the quadrics $Q_{\alpha + \lambda \beta}$ on spaces of this pencil are

$$(\alpha x^2) + \lambda (\beta x^2) = 0.$$

It follows that the quadrics $Q_{\alpha + \lambda \beta}$ cut out on S a pencil of conics, Φ say. The three points of intersection with S of its trisecant basic R^4 are apolar to all quadrics $Q_{\alpha+\lambda\beta}$, by theorem c′). They must therefore be apolar to all conics Φ, and so must be the vertices of the diagonal triangle of the pencil Φ.

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* Sturm, *loc. cit.*

If a basic R^4 touches S it touches every space of the pencil on S. Hence:

The four basic R^4's tangent to S touch at the base-points of the pencil Φ.

The base-planes and spaces meet S in a configuration, Δ say, made up of twenty points and fifteen lines, four points on a line, three lines on a point. Δ is, by the Veronese "Perspective Pyramid Theorem," the figure of three triangles in continued perspective, taken with their axes of perspection,—these three axes meet in a point. Since the lines (on a space $\alpha+\lambda\beta$) 123 and 456 are polar lines in the quadric $Q_{\alpha+\lambda\beta}$, it follows that the points 123 and 456 are apolar to every conic Φ. These results easily give the following theorem:

The six conics defined by the six configurations B of Δ are in a pencil Φ, the base-points of which are the points of tangency of tangent basic R^4's to S. The vertices of the triangle apolar to Φ are the points of intersection with S of its trisecant basic R^4.[*]

Basic R^4's meeting a line, p, lie on a 2-way, P, of order 9. Any space, α, on p meets P in a nonic curve, the line p and an octavic curve P' in α. Any plane S on p and in α meets P' in eight points. Now P' meets p in two points, the two points on p in which basic R^4's touch α. Hence there are six basic R^4's that meet a line p on S and meet S at some other point not on p. Therefore:

The locus of points of intersection with a plane, S, of its bisecant basic R^4's is a sextic curve, Σ.

Σ has obviously three nodes, at the points of intersection with S of its trisecant basic R^4. It can in general have no further nodes, since this would require two basic R^4's on a point of S. Its genus is therefore 7. It contains the twenty points of the configuration Δ, conics of the pencils on 123 and 456 and on the points where S meets these planes, forming a basic R^4 bisecant to S. It also contains the base-points of the pencil Φ, and touches at these points the tangents to basic R^4's through them.

Consider the set of basic R^4's as projected from a base-line, 12 say. They are projected by quadric spreads with 12 as a nodal line, and project into conics on S on 123, 124, 125, 126. Points of intersection with S of bisecant basic R^4's must be on conics of this pencil. They must further be apolar to all conics Φ. Hence:

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[*] Carver: *Trans. Am. Math. Soc.*, Vol. VI, No. 4, p. 543.

Points of intersection with a plane, S, of its bisecant basic R^4's are partners in the Cremona involution determined by the pencil Φ associated with the configuration Δ on S. They are also on conics of fifteen pencils on 4-points of Δ like 123, 124, 125, 126.

We saw in Part II that the locus of pairs of points determined in this way is a sextic curve. The involution determined by conics Φ leaves this sextic unaltered. It carries the lines of Δ into conics on L, M, N, the diagonal 3-point of Φ. It carries 123 into 456, etc., as we have seen, thus leaving the points of Δ, as a whole, unaltered. Hence:

The points 123, 124, 125, 126, *L, M, N are on a conic.*

This is the conic into which the basic R^4 trisecant to S is projected from 12. Δ determines in this way a configuration of conics with the same set of points and the same incidence conditions.

We also proved in Part II that the locus of joins of corresponding points on the sextic Σ is a line-quartic, W say. W touches the six lines of both bases Φ and Φ', the second generating pencil Φ' being conics on 123, 124, 125, 126. Since W is symmetrically related to Δ, it touches all the twenty lines of Δ. Since 123 and 456 are corresponding points of Σ, W touches the ten lines 123/456. It obviously touches the three lines of the diagonal triangle of Φ. But it is more specially related to these lines, as we shall see.

The configuration Δ_1 dual to Δ determines a range of conics Φ_1. Here we have a line-sextic Σ_1 and a point-quartic W_1. The configuration Δ_1 may be considered as the section by a plane of the complete 6-point in three dimensions. Here W_1 is the section by S of the Weddle surface with nodes at the six points. The diagonal 3-point of the range Φ_1 is the three points in which the R^3 on the six points in space meet the plane. The properties of the plane section of a Weddle quartic* may be here stated for the sextic Σ and the quartic W:

The line-quartic W determined by basic R^4's on a plane S touches the three lines of the diagonal triangle of the pencil Φ in three points on a line. The two tangents to W from a point of the diagonal 3-point, besides the two diagonal lines on the point, are apolar to these two lines. The configuration Δ is not unique on W, there being an infinity of such configurations whose lines touch W and whose points are on Σ.

The first part of this theorem may be verified easily from the form of the equation given in Part II. The specialization here is that one of the conics

$$(\alpha x)^2 + \lambda (\beta x)^2 = 0$$

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* Morley and Conner, *loc. cit.*

is on the reference 3-point. Let this be $(\alpha x)^2$. We have, then,

$$\alpha_{11} = \alpha_{22} = \alpha_{33} = 0,$$

and equation (13), Part II, becomes

$$\Sigma\, \alpha_{12}\, \beta_{33}\, \xi_1\, \xi_2\, (\xi_1^2 - \xi_2^2) - 2\, \Sigma\, (\alpha_{12}\, \beta_{13} - \alpha_{13}\, \beta_{12})\, \xi_1^2\, \xi_2\, \xi_3 = 0,$$

from which the first part of the above theorem is an easy inference.

Let us return for a moment to the system of quadrics on six points of space. It is three linear conditions on a quadric to touch a given plane at a given point. Hence one quadric on the six points touches a plane, S, at a given point. The two lines cut out of this quadric by S are partners in the Cremona line-involution determined by the range Φ_1 on S. At a point, a, of the quartic W_1, the quadric tangent to S is a cone with vertex at a. The two lines cut out of this cone by S are lines of the sextic Σ_1. One quadric on the six points contains an arbitrarily chosen line of space. If we require this quadric to have a node, the line lies in a complex. We have, therefore:

Generators of quadric cones on six points of space are in a sextic complex. The complex curve on any plane, S, is the sextic Σ_1. This complex contains the (1, 3) *congruence of bisecants to the R^3 on the six points doubly.*[*]

The range Φ_1 of the configuration Δ_1 determines a conjugate 3-fold system of conics to which all conics Φ_1 are apolar. If the plane α is mapped by this scheme on a Steiner quartic surface, Ω, the lines 123, 456, being conjugate lines in the range, will go by this mapping scheme into a tangent plane section of Ω, touching at 123/456. We get in this way ten planes and fifteen points. If we indicate the points of Δ_1 by two-figure symbols, the incidence relations of these ten planes and fifteen points may be exhibited in the two diagrams:

x	46	62	24
35	12	14	16
51	32	34	36
13	52	54	56

135	x	x	x
x	146	162	124
x	346	362	324
x	546	562	524.

135 in the second diagram stands for 135/246, etc. A point in the left-hand diagram is incident with the planes in the right-hand diagram standing in its row and column, except the one in its position, and similarly for the planes. But this is exactly the scheme of nodes and tropes of a 15-nodal quartic surface. Therefore:

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[*] Sturm: "Die Lehre von den Geom. Verwandtschaften," III, p. 409.

Conics apolar to the range Φ_1 of Δ_1 map the plane into a Steiner quartic surface Ω. The points of Δ_1 go into fifteen points of Ω. The lines 123, 456, *etc., go into ten tangent plane sections of Ω. The points and planes thus obtained form the configuration of nodes and tropes of a* 15-*nodal quartic surface, fifteen points lying by sixes on ten conics.*

§ 33. *Complex of Lines Bisecant to Basic R^4's.*

We proved at the beginning of Part IV that two points in which a basic R^4 meets a space α are points of an R^3 of a basic system in α and apolar to Q_α. At the end of Part III we showed that the lines joining pairs of points thus determined are in a quartic complex. Hence:

Lines bisecant to basic R^4's are in a quartic complex.

It is obvious that these lines may also be defined as the lines met by quadrics $Q_{\alpha+\lambda\beta+\mu\gamma}$ of the net of spaces on them in pairs of an involution. This property makes the finding of the equation of the four-dimensional complex easy by methods similar to those employed in Part III. The following is also obvious:

The complex curve on any plane, S, is the quartic W.

We see from equation (28), Part III, that if $p_{12}=p_{13}=p_{14}=0$, involving $\pi_{34}=\pi_{42}=\pi_{23}=0$, or $p_{12}=p_{23}=p_{31}=0$, involving $\pi_{34}=\pi_{14}=\pi_{24}=0$, the equation of the complex is satisfied. Hence the complex (28) contains every line on a base-point or on a base-plane. Since our complex is symmetrically related to the configuration Γ in α, we have:

The section by a space α of the complex of lines bisecant to basic R^4's contains singly every line on the planes of the configuration Γ in α, and every line on its points. This amounts to saying: The complex of bisecants to basic R^4's contains, in general singly, every line meeting a base-line, and every line lying on a base-space.

This is also evident directly. A line on a base-space has a unique bisecant basic R^3 of the system in that space, and hence a degenerate bisecant basic R^4. A line meeting 12 meets on R^3 of the basic system on 3456, the R^3 on the point where it meets 3456, and hence has a degenerate bisecant basic R^4. A similar argument shows that our quartic complex contains every line meeting opposite planes 123/456, etc.

It was shown by Sturm that the complex of tangents to basic R^3's contains lines on base-points and base-planes doubly. The section of our complex with the sextic complex determined by 16, 26, 36, 46, 56, as base, leaving out of account the above-mentioned degenerate congruences, is a congruence of order $6.4 - 5.2 = 14$, and of class $6.4 - 10.2 = 4$. This is the $(14, 4)$ congruence of tangents to basic R^4's.

§ 34. *An* 11-*ic* 2-*way Determined by Seven Points.*

Consider any plane, S, and a space, α, on it. Basic R^4's bisecant to S meet α in two other points, and the lines joining these pairs of points form the complex cone on α of bisecants to basic R^4's through M, the pole of S as to Q_a. The locus of these two extra points of intersection is a curve in S with a triple point at M, since the trisecant basic R^4 to S counts as three bisecant R^4's, and this R^4 passes through M by theorem c′). This locus is met by a plane in α through M in eight points on the cone proper, and in three points at M. It is obviously the locus of points in α from which 1, 2, 3, 4, 5, 6, M are projected into seven points of an R^3, since an R^4 projects into an R^3 from any point on it. Hence:

The locus of points in S_4 *from which seven points project into seven points of an* R^3 *is a* 2-*way*, Ψ *of order* 11, *with triple points at each of the seven points.*

The surface Ψ shows a close analogy with the Weddle surface in S_3. Let us project it from one of its triple points, M say, into a space β. Consider any point, x, of the surface. If we project 1, 2, 3, 4, 5, 6 from x on a space γ through M, the points 1γ, 2γ, 3γ, 4γ, 5γ, 6γ* are on an R^3 with M in γ, and are projected from M by six lines of a quadric cone, V, in γ. Now the lines $x1$, $x2$, etc., project from M into β into the lines $x_\beta 1_\beta$, $x_\beta 2_\beta$, etc. Since $x1$, $x2$, etc., meet γ in points of the R^3 $1\gamma, \ldots, M$, and hence in points of V, $x_\beta 1_\beta$ must meet β_γ in points of the conic V_β. In other words, 1_β, 2_β, etc., project from x_β on the plane β_γ into six points of a conic. Hence:

The surface Ψ *projects from any one of its triple points into a Weddle surface taken twice.*

The 11-ic section of Ψ on our space through M projects into a plane section of this Weddle surface. We have, then:

The section by a plane in α *of the cone of the complex of bisecants to basic* R^4's *may be regarded as a plane section of a Weddle surface*, a theorem which we have met in another connection.

The 11-ic curve which is the section of the surface Ψ by a space α on M, is the section (in α) of the Weddle surface with nodes at 16, 26, 36, 46, 56, M, in which the R^n meets the polar S_{n-1} of a as to Q. It is $2n+1$ linear conditions on a C_{n-1}^2 to contain a given R^n. There are then

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* 1γ is meant to indicate the point into which 1 projects.

$$\tfrac{1}{2}n(n+3)-2n-1+1=\tfrac{1}{2}n(n-1)$$

linearly independent C_{n-1}^2's on a given R^n. It is, then, $\frac{1}{2}n(n-1)$ conditions on the system of quadrics to be apolar to Q, and hence it is $\frac{1}{2}n(n-1)$ conditions on the R^n to be orthic to Q.

Basic R^n's are projected from a base-point by a system of ∞^{n-2} C_2^{n-1}'s. There are ∞^1 basic R^n's on each of these cones, one through every point of the cone. These C_2^{n-1}'s meet α in a basic system of R^{n-1}'s on an orthic base of Γ^n. We mean by an "orthic base" of Γ^n a set of $n+1$ points of Γ^n such that every C_{n-2}^2 on them is apolar to Q_α. There are $n+2$ such orthic bases in Γ^n. All points of Γ^n whose names contain one symbol form an orthic base, as 12, 13, 14, ... Every R^{n-1} on an orthic base is orthic to Q_α. The following theorems are sufficiently evident from what we have said in discussing the 3- and 4-dimensional cases.

Sets of points $\iota_n^{(n)}$ in α are n-points apolar to Q_α, and inscribed in R^n's of the basic system on any orthic base of Γ^n.

All loci determined in α by basic R^n's are completely determined by Q_α and an orthic base of Γ^n.

Choosing as reference points in α n points of an orthic base of Γ^n, with the $(n+1)$-st point as unit point, a basic R^n on a point x may be given parametrically,

$$y_i=\frac{x_i}{1-x_i t}, \quad (i=1,\ \ldots,\ n), \tag{1}$$

the point x having the parameter value $t=0$. We indicate the quadric Q_α by

$$Q=0.$$

The conditions on the coefficients of Q may easily be worked out. The polar S_{n-2} of x as to Q is

$$\Sigma y_i \frac{\partial Q}{\partial x_i}=0. \tag{2}$$

(2) cuts out of the R^{n-1} (1) the $n-1$ points which, taken with x, form a set $\iota_n^{(n)}$ in α. The $(n-1)$-ic equation in t giving the parameters of these $n-1$ points may be obtained by substitution of (1) in (2). It is

$$x_1 x_2 \ldots . x_n L t^{n-1}-f_n t^{n-2}+\ldots .+(-)^{n-2} f_2 t+(-)^{n-1} . 2Q=0, \tag{3}$$

where
$$f_{i+1} \equiv \Sigma x_1 \ldots . x_i \left(\frac{\partial Q}{\partial x_1} + \ldots . + \frac{\partial Q}{\partial x_i} \right),$$
and
$$L \equiv \Sigma \frac{\partial Q}{\partial x_i}.$$
L is the polar S_{n-2} of the unit point as to Q. The spread f_{i+1} is a C^{i+1}_{n-2} on the $C^{(i+1)!}_{n-i-1}$, the locus of points of contact of basic R^n's having $(i+1)$-point contact with α.

The vanishing of an invariant of (3) of degree i and weight w evidently gives as the locus of the point x a C^{w+2i}_{n-2}. Since
$$w = \tfrac{1}{2} i (n - 1),$$
we have the theorem:

The locus of points x in α such that the $n-1$ points besides x of the set $\iota^{(n)}_n$ determined by x are $n-1$ points on the basic R^{n-1} on x and an orthic base in Γ^n, and, as points on this R^{n-1} as a binary support, have a vanishing invariant I_i, is a C_{n-2}, of order $\tfrac{1}{2} i (n+3)$.

The n points of a set $\iota^{(n)}_n$ considered as a set of the basic R^n are projectively equivalent to the n points considered as points of the corresponding R^{n-1}, since the R^{n-1} is a projection of the R^n, and these n points are self-corresponding. Hence (3) multiplied by t,
$$x_1 \ldots . x_n L t^n - f_n t^{n-1} + \ldots . = 0, \tag{4}$$
represents to within a collineation on the R^n the n points of intersection with α of the basic R^n on x. The vanishing of an invariant of (4) must, then, carry with it the vanishing of the corresponding invariant of the n points on the basic R^n. An invariant of (4) of degree i and weight w is of degree $w+i$ in the x's. Since $w = \tfrac{1}{2} n i$, we have
$$w + i = \tfrac{1}{2} i (n+2).$$
Hence:

Basic R^n's meeting α in sets $\iota^{(n)}_n$ for which an invariant I_i (on the R^n) vanishes meet α in points of a C_{n-2} of order $\tfrac{1}{2} i (n+2)$.

§ 37. *A Cubic Spread.*

The C_{n-2}'s, $Q_{\alpha + \lambda\beta}$, on S_{n-1}'s of a pencil,
$$(\alpha x) + \lambda (\beta x) = 0,$$
lie on the C^3_{n-1}
$$\begin{vmatrix} (\alpha x^2), & (\beta x^2) \\ (\alpha x), & (\beta x) \end{vmatrix} \equiv \Sigma \pi_{ij} x_i x_j (x_i - x_j) = 0, \tag{5}$$

where $\pi_{ij}=\alpha_i\beta_j-\alpha_j\beta_i$ are the coördinates of the S_{n-2} common to α and β. Now the basic R^n on x passes through x with the tangent $p_{ij}=x_ix_j(x_i-x_j)$. Hence:

The locus of points of tangency with basic R^n's of tangents meeting an S_{n-2}, π, is the C^3_{n-1} (5).

This locus contains the S_{n-2} singly.

Consider two S_{n-2}'s that have an S_{n-3} in common. The two C^3_{n-1}'s determined by the two S_{n-2}'s meet in a C^9_{n-2}, out of which the C^2_{n-2} Q_α on the S_{n-1}, α, common to the two S_{n-2}'s, factors. Hence:

The locus of points of tangency with basic R^n's of tangents meeting an S_{n-3} is a C^7_{n-2}.

Consider an S_{n-2} and an S_{n-3} which have an S_{n-4} in common. There is an S_{n-1}, α, containing both S_{n-2} and S_{n-3}. Tangents to basic R^n's in α are tangents to basic R^{n-1}'s on an orthic base in α at points of Q_α. Lines tangent to basic R^n's and meeting S_{n-3} in α are tangents to basic R^{n-1}'s in α which meet S_{n-3} and have their points of tangency on Q_α. The locus of these points of tangency is the intersection of a C^3_{n-2} with Q_α, — a C^6_{n-3}. The C^3_{n-1} and the C^7_{n-2} determined in S_n by S_{n-2} and S_{n-3} respectively intersect in a C^{21}_{n-3}, out of which the above C^6_{n-3} factors. Hence:

The locus of points of tangency with basic R^n's of tangents meeting an S_{n-4} is a C^{15}_{n-3}.

We can now deduce the general law. Call the order of the C_{n-r+1} of points of tangency with basic R^n's of tangents meeting an S_{n-r}, m_r. Consider an S_{n-2} and an S_{n-r+1} which have an S_{n-r} in common. There is an S_{n-1}, α, containing both S_{n-r+1} and S_{n-2}. For an S_{n-1} containing $n-r+1$ points of S_{n-r}, one further point of S_{n-r+1}, and $r-2$ further points of S_{n-2}, contains both spaces, and is on n points, —enough to fix it. Lines tangent to basic R^n's, in α and meeting S_{n-r+1}, are lines tangent to basic R^{n-1}'s in α and meeting S_{n-r+1}, and having their points of tangency on Q_α. The locus of these points of tangency is the intersection of a C_{n-r+2} of order m_{r-2} with Q_α, —a C_{n-r+1} of order $2m_{r-2}$. The C^3_{n-1} and the C_{n-r+2} of order m_{r-1} determined by S_{n-2} and S_{n-r+1} intersect in a C_{n-r+1} of order $3m_{r-1}$ out of which the above C_{n-r+1} of order $2m_{r-2}$ factors. This gives the recurrence formula

$$m_r=3m_{r-1}-2m_{r-2},$$

whence

$$m_r=2^r-1.$$

Hence:

The locus of points of tangency with basic R^n's of tangents meeting an S_{n-r} is a C_{n-r+1}, of order 2^r-1.

In particular we have:

The locus of points of tangency with basic R^n's of tangents on a point (S_0) is a C_1, a curve of order 2^n-1.

This curve is projected from a point of it by a cone C_2 of order 2^n-2. Hence:

Lines tangent to basic R^n's and on a point lie on a C_2-cone of order 2^n-2.

§ 38. *A Generalization of the Weddle Surface.*

The argument of § 34, Part IV, obviously holds in n dimensions. We have, then, that the locus, Ψ^n, of points from which $n+3$ points in S_n project into $n+3$ points of an R^{n-1} is a C_2 with these properties:

1) *It has $(n-1)$-fold points at the $n+3$ points.*

2) *It projects from one of these points into a Ψ^{n-1} taken twice.*

If m_n is the order of Ψ^n, we have the recurrence formula

$$m_n = n - 1 + 2\,m_{n-1},$$

whence

$$m_n = 2^n - n - 1.$$

Hence:

The locus of points in S_n from which $n+3$ points project into $n+3$ points of an R^{n-1} is a $C_2^{m_n}$, where m_n has the above value.

This locus has properties closely analogous to those of the Weddle surface.

We have also:

Lines bisecant to basic R^n's and on a point, a, in S_n lie on a C_3 of order m_{n-1}.

The points of intersection with these lines of their bisecant basic R^n's are on the C_2, Ψ^n.

§ 39. *Some Enumerative Theorems.*

We can find the order of the C_{n-2} in which basic R^n's tangent to α meet α again by finding the order of its C_{n-3} of intersection with an S_{n-2} of Γ^n. A basic R^n cannot meet a base S_{n-1} without degenerating, and hence we need consider only the degenerate basic R^n's. And we evidently need consider only the basic R^n's degenerating into lines and R^{n-1}'s, and into conics and R^{n-2}'s. Let us assume that the order of the C_{n-3} of further intersection with an S_{n-2} of tangent basic R^{n-1}'s in S_{n-1} is m_{n-1}. Basic R^{n-1}'s in the S_{n-1} cutting out the S_{n-2} from Γ^n determine a $C^{m_{n-1}}_{n-3}$ on S_{n-2}. All S_{n-3}'s on the S_{n-2} in Γ^n count doubly in $C^{m_n}_{n-2}$, since two conics of the pencil in the plane opposite to a base S_{n-2} touch α, and either of these conics, taken with a basic R^{n-2} on the opposite S_{n-2}, is an R^n, satisfying our conditions. Since there are n S_{n-3}'s on an S_{n-2} of Γ^n, we have the recurrence formula

$$m_n = 2n + m_{n-1},$$

whence

$$m_n = (n+3)(n-2).$$

This gives us the theorem:

Basic R^n's tangent to an S_{n-1}, α, meet α again in a $C^{(n+3)(n-2)}_{n-2}$.

This C_{n-2} is the discriminant of (4). It touches Q_α along the C^6_{n-3} of points of osculation of basic R^n's with α and cuts out of Q_α the points of contact of bitangent basic R^n's. Hence:

Points of contact with α of bitangent basic R^n's lie on a $C^{2(n+4)(n-3)}_{n-3}$.

It may be proved as in Part IV that lines joining corresponding points of this C_{n-3} are lines of Q_α.

We now ask for the order of the C_{n-p} in which basic R^n's having p-point contact with α meet α again. Call it m^n_p. We can find the order of C_{n-p} by finding the order of the C_{n-p+1} in which it meets an S_{n-2} of Γ^n. The S_{n-p-1}'s of Γ^n count $p!$-fold in the locus, since they are cut out by base S_{n-p}'s, and $p!$ basic R^p's of the systems on the opposite S_p's have p-point contact with α. There are $\binom{n}{n-p+1}$ S_{n-p-1}'s on an S_{n-2} of Γ^n. There are on S_i's of Γ^n, where

$$n-p-1 < i < n-2,$$

spreads belonging to our locus, but their dimension is less than $n-p-1$, and we need not take them into account. On the S_{n-1} cutting out our S_{n-2} we have a basic system of R^{n-1}'s, and these determine a C^t_{n-p-1} on S_{n-2}, where $t = m^{n-1}_p$. Hence we have the recurrence formula

$$m^n_p = \binom{n}{n-p+1} p! + m^{n-1}_p,$$

whence

$$m^n_p = p!\left[\binom{n+1}{p} - p - 1\right]. \tag{6}$$

Hence:

Basic R^n's having p-point contact with α meet α again in a C^t_{n-p}, where $t = m^n_p$ as given in (6).

It is $n-i-1$ conditions on a curve in S_n to meet an S_i. If the locus of basic R^n's meeting an S_{n-p} is a C_{n-p+1} of order x^n_p, an S_{p-1} meets this locus in x^n_p points, and there are x^n_p basic R^n's meeting an S_{n-p} and an S_{p-1} by the theorem of Sturm. Hence the locus of basic R^n's meeting an S_{p-1} is met by an S_{n-p} in x^n_p points, and the order of the latter locus is x^n_p. The locus of basic R^n's meeting a C^{μ}_{n-p} is met by an S_{p-1} in the μx^n_p points defined on S_{p-1} by the basic R^n's meeting S_{p-1} and C^{μ}_{n-p}. The order of the locus of basic R^n's meeting C^{μ}_{n-p} is therefore μx^n_p, provided the C^{μ}_{n-p} does not meet a base S_i where $i<p$. We have seen that basic R^n's have p-point contact with α at points of a $C^{p!}_{n-p}$ and meet again in points of a C_{n-p} of order m^n_p. The $C^{p!}_{n-p}$ does not meet a base S_i where $i<p$, since an R^i cannot have p-point contact with α if $i<p$. The locus of basic R^n's meeting $C^{p!}_{n-p}$ is a C_{n-p+1} which meets α in $C^{p!}_{n-p}$ taken p times and in the C_{n-p} of order m^n_p. Hence:

$$x^n_p = \frac{p \cdot p! + m^n_p}{p!} = \binom{n+1}{p} - 1. \tag{7}$$

Therefore:

Basic R^n's meeting an S_{n-p} lie on a C_{n-p+1} of order x^n_p.

There are x^n_p basic R^n's that meet an S_{n-p} and an S_{p-1}.

Basic R^n's meeting an S_{p-1} lie on a C_p of order x^n_p.

The identity involved in these theorems,

$$x^n_p = x^n_{n-p+1},$$

is evident from the form of (7).

The following theorems are easily verified by the methods of Part IV:

Basic R^n's that touch a C^m_{n-1} meet it again in a C^{μ}_{n-3}, where

$$\mu = m(m+1)\left[\frac{(n-1)(n+2)}{2}m - 2\right].$$

Basic R^n's bitangent to a C^m_{n-1} touch at points of a C^{ν}_{n-3}, where

$$\nu = m(m+1)\left[\frac{(n-1)(n+2)m(m+1)}{2} - 4m - 6\right].$$

The multiplicity of a base S_r in the C_{n-p+1} of basic R^n's meeting an S_{n-p} is $\binom{n-r}{p}$.

By counting of constants we should expect to find a finite number of basic R^n's on a basic C^2_{n-1}. An R^n in S_n is determined by $(n+1)^2-4$ constants. It is $2n+1$ conditions on the R^n to lie on a C^2_{n-1}. It is $n-2$ conditions on a curve lying on a C_{n-1} to pass through a given point of the C_{n-1}. Hence:

An R^n *on a* C^2_{n-1} *in* S_n *is determined by*

$$\frac{(n+1)^2-4-2n+1}{n-2}=n+2$$

points, — a base, — to within a finite number of positions.

By a theorem of Part I, basic R^n's having $(n-1)$-point contact with a C^2_{n-1} touch at points of a $C_1^{n!}$. If the C^2_{n-1} is basic, this $C_1^{n!}$ exists, but can be nothing but basic R^n's, since an R^n having $(n-1)$-point contact with the C^2_{n-2} has $n+2+n-1=2n+1$ points in common with it, and must lie on it. Hence:

There are $(n-1)!$ *basic* R^n*'s on a basic* C^2_{n-1}.

Basic R^n's meeting an S_{n-2} r times lie on a C_{n-r}; let us assume that its order is τ^n_r. We obtain a formula for τ^n_r by looking for the C_{n-r-1} in which the C_{n-r} meets a base S_{n-1}. The given S_{n-2} meets a base S_{n-1} in an S_{n-3}; basic R^{n-1}'s on S_{n-1} meeting S_{n-3} r times lie on a C_{n-r-1} of order τ^{n-1}_r. Consider a base S_{n-i} and the opposite S_i. S_{n-2} meets S_i in an S_{i-2}. One basic R^i meets S_{i-2} $i-1$ times. S_{n-2} meets the base S_{n-i} in an S_{n-i-2}. The basic R^{n-i}'s on S_{n-i} meeting this S_{n-i-2} $r-i+1$ times (enough to make S_{n-2} meet R^{n-i} taken with the R^i in the opposite S_i r times) lie on a C_{n-r-1} of order τ^{n-i}_{r-i+1}. There are $\binom{n}{i-1}$ base S_{n-i}'s on a base S_{n-1}. This gives the formula

$$^*\tau^n_r=\tau^{n-1}_r+\binom{n}{1}\tau^{n-2}_{r-1}+\binom{n}{2}\tau^{n-3}_{r-2}+\cdots\cdot+\binom{n}{r}\tau^{n-r-1}_0, \tag{8}$$

by which τ^n_r may be calculated in any particular case. The following values of special τ's are in accordance with the facts:

$$\tau^n_0=1,\quad \tau^n_n=0\,(n\neq 0),\quad \tau^0_0=1,\quad \tau^m_n=0\,(n>m),\quad \tau^n_{n-1}=n.$$

The following general formula for τ^{n+1}_{n-r} may be verified by induction from (8):

$$\tau^{n+1}_{n-r}=\frac{r!\,(n+1)-(r+1)!\sum\limits_{i=1}^{r}\frac{1}{i}+\sum\limits_{i=1}^{r}(-)^{i+1}(r!)^2(i-1)!\,(r-i+1)^{n+1}}{(r!)^2}.$$

Basic R^n's meeting r S_{n-2}'s lie on a C_{n-r}; let us assume that its order is σ^n_r. An argument similar to the above gives us a formula by means of which σ^n_r may be calculated. Let us examine the C_{n-r-1} in which our C_{n-r} meets a base S_{n-1}. The r S_{n-2}'s meet S_{n-1} in r S_{n-3}'s, and these give a C_{n-r-1} of order σ^{n-1}_r which belongs to our locus. Consider a base S_{n-i} and the opposite S_i. $i-1$ S_{n-2}'s meet S_i in $i-1$ S_{i-2}'s. These can be chosen from the r S_{n-2}'s in $\binom{r}{i-1}$ ways. There are σ^i_{i-1}/i R^i's in S_i meeting these S_{i-2}'s. The remaining $r-i+1$ S_{n-2}'s meet S_{n-i} in $r-i+1$ S_{n-i-2}'s. Basic R^{n-i}'s on S_{n-i} meeting these S_{n-i-2}'s lie on a C_{n-r-1} of order σ^{n-i}_{r-i+1}. There being $\binom{n}{i-1}$ S_{n-i}'s on a base S_{n-1}, we have the formula

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* It is a remarkable fact that the number τ^{n+1}_{n-1} is the genus of the curve of intersection of $n-1$ C^2_{n-1}'s in S_r.

$$\sigma^n_r = \sigma^{n-1}_r + \binom{n}{1}\binom{r}{1}\sigma^{n-2}_{r-1}\sigma^2_1/2 + \binom{n}{2}\binom{r}{2}\sigma^{n-3}_{r-2}\sigma^3_2/3 + \dots \\ + \binom{n}{r}\binom{r}{r}\sigma^{n-r-1}_0\sigma^{r+1}_r/r+1. \tag{9}$$

The following values of special σ's are in accordance with the facts:

$$\sigma^n_0 = 1,\ \ \sigma^n_n = 0\,(n \neq 0),\ \ \sigma^0_0 = 1,\ \ \sigma^m_n = 0\,(n > m),\ \ \sigma^n_{n-1} = n\,\sigma^n_{n-2}.$$

It is easy to calculate any special τ or σ from (8) or (9) taken with the table of special values. Suppose, for example, we require the value of σ^4_3. In the first place,

$$\sigma^4_3 = 4\,\sigma^4_2.$$

From (9) we have

$$\sigma^4_2 = \sigma^3_2 + \binom{4}{1}\binom{2}{1}\sigma^2_1\sigma^2_1/2 + \binom{4}{2}\binom{2}{2}\sigma^1_0\sigma^3_2/3.$$

We have also

$$\sigma^2_1 = 2\,\sigma^2_0 = 2,\ \ \sigma^3_2 = 3\,\sigma^3_1 = 3\left[\sigma^2_1 + \binom{3}{1}\binom{1}{1}\sigma^1_0\sigma^2_1/2\right] = 15,$$

whence

$$\sigma^4_2 = 61,$$

a number which we found in Part III. This gives

$$\sigma^4_3 = 4\,.\,61 = 244.$$

JOHNS HOPKINS UNIVERSITY, *March*, 1909.

Surfaces Invariant under Infinite Discontinuous Birational Groups Defined by Line Congruences.

BY VIRGIL SNYDER.

A family of quadric surfaces containing one variable parameter will define a congruence of lines having the envelope for focal surface. When the parameter enters rationally to the power n, the congruence will be of order $2n$ and of class $6n$; the focal surface will be of order $4(n-1)$. Each quadric will touch it along a space quartic of the first kind, defined as the limiting intersection of two quadrics which approach coincidence.

1. When the two systems of generators are rationally separable, the congruence breaks up into two, of the form $(n, 3n)$, having the same focal surface. These surfaces are interesting from the fact that the congruences will define upon them an infinite discontinuous group of birational transformations which leave the surface invariant.*

Let

$$\frac{\sum_{i=0}^{m} a_i \lambda^{m-i} \mu^i}{\sum_{k=0}^{n} b_k \lambda^{n-k} \mu^k} = \frac{\sum_{i=0}^{m} c_i \lambda^{m-i} \mu^i}{\sum_{k=0}^{n} d_k \lambda^{n-k} \mu^k} = \sigma, \tag{1}$$

a_i, b_k, c_i, d_k being linear expressions in four homogeneous variables (x), be the equations of a line. As $\lambda : \mu$ and σ take various values, this line will define a congruence which we shall call the σ congruence. Similarly, by interchanging Σc, Σb, a new congruence τ is defined.

These two congruences are arranged on a system of quadrics, and have the same focal surface ϕ. The cones contained in this family evidently belong to both congruences, and their vertices are singular points on ϕ. Should a cone further break up into a pair of planes π_1, π_2, the corresponding lines of σ would belong to two pencils $(0_1, \pi_1)$, $(0_2, \pi_2)$, and the lines of τ to the pencils $(0_1, \pi_2)$, $(0_2, \pi_1)$, wherein 0_1, 0_2 are two points on the line of intersection of the two planes. The points 0_1, 0_2 are singular points on ϕ, and π_1, π_2 are double planes, touching ϕ along conics.

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* Cf. "Infinite Discontinuous Groups of Birational Transformations Which Leave Certain Surfaces Invariant," *Transactions American Math. Society*, Vol. XI (1910), pp. 15–24.

Through any point $P_1 \equiv (x_1)$ on ϕ pass a line of σ and a line of τ which touch the surface at that point.

2. When the coordinates of P_1 are substituted in (1), the binary quartic in $\lambda : \mu$ will have a double root which can be expressed rationally in terms of (x_1). The corresponding $\lambda_1 : \mu_1$ determines the single quadric $Q(\lambda_1)$ which touches ϕ at P_1. The curve of contact c_4 is defined by $Q(\lambda_1) = 0$, $\frac{\partial Q}{\partial \lambda_1} = 0$. A line of σ touching ϕ at P_1 will therefore touch it again at the second point of intersection with $\frac{\partial Q}{\partial \lambda_1}$. Since the coordinates of one point of intersection are given, those of the other point $P_2 = (x_2)$ are expressed rationally in terms of (x_1). Had we started with P_2, we could express the coordinates of P_1 rationally in terms of (x_2).

The operation of interchanging the points of contact of any line of σ with the focal surface ϕ is a birational transformation of order 2 which leaves ϕ invariant.

This property is true for a large category of congruences that cannot be expressed in the form (1), but is not true for those defined by a family of quadrics in which the two systems of generators are not rationally separable, for, corresponding to a point P_1 we have two points P_2, one on each generator of the quadric passing through P_1.*

In the same way, the congruence τ will define a birational transformation of order 2. Let these be denoted by (Σ), (T).

3. The coordinates of a point on c_4 can be expressed in terms of elliptic functions in such a way that the parameters u of four points lying in a plane will have a sum conjugate to zero. A quadric can be passed through any c_4 and one bisecant. Let u_1, u_2 be two points such that $u_1 + u_2 = k$ is not commen-

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* A congruence of this kind can be obtained from the cubic variety Γ in space of four dimensions, which contains a plane, but cannot be generated by trilinear systems. Let a pencil of spaces have the plane for basis. Each space will cut Γ in a residual quadric, one passing through each point. If the quadrics be projected from a point on Γ into any space not passing through the center of projection, the result is a family of index 2, $\lambda^2 u + \lambda \mu v + \mu^2 w = 0$. The apparent contour is the envelope of this system, and complete focal surface of the congruence. The surface $4uw - v^2 = 0$ has eight double points, and is of class 20. The vertices of the cones contained in the family of quadrics are not in this case double points of the focal surface.

surable with the periods of the elliptic functions concerned. The planes passed through this line will contain a generator of the other system such that the sum of the two parameters will be conjugate to $-k$. In particular, if u_1 define P_1, then P_2 is expressed by $k-u_1$; from P_2 a generator of τ meets c_4 again in u_1-2k. If the operation (ΣT) be repeated, the point P_1 can never be reached after a finite number of transformations, since k was chosen incommensurable with the periods. If the constants be so chosen for one value of λ, it insures that (ΣT) is of infinite order.

4. The order of the discriminant is in general $2(m+n-1)$, but will contain as factor any letter common to the coefficient of λ^{m+n}, $\lambda\mu^{m+n-1}$, and similarly for $\lambda\mu^{m+n-1}$, μ^{m+n}. Thus the order of ϕ is in general of order a multiple of 4, but may be reduced by one unit if $c_0=0$, $b_0=0$, $a_1=a_0$; by two units if $d_1=a_1$; and by three units if also $b_n=c_m=d_n=0$; hence:

Surfaces ϕ of any order except 5 can be constructed on which the operation (ΣT) *defines an infinite discontinuous group of birational transformations under which ϕ remains invariant.*

5. Among the congruences expressible in the form (1) are the Cremona congruences, obtained by connecting corresponding points of two planes in (1, 1) correspondence.*

The general theory will not be discussed further, but a few interesting applications will be made to well-known congruences.

Congruences of Order 2.

6. Given the three trilinear systems in space of four dimensions S_4,

$$\alpha u_1+\beta u_2+\gamma u_3=0,\quad \alpha v_1+\beta v_2+\gamma v_3=0,\quad \alpha w_1+\beta w_2+\gamma w_3=0.$$

They will generate the cubic variety Γ defined by

$$\Gamma\equiv\begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}=0.$$

Without restricting the system, we may put:

$$\begin{aligned} &u_1=x_1, \quad v_1=x_4, && w_1=b_1x_1+b_2x_2+b_3x_3=b,\\ &u_2=x_2, \quad v_2=x_5, && w_2=c_1x_1+c_2x_2+c_4x_4=c,\\ &u_3=x_3, \quad v_3=a_1x_1+a_2x_2+a_3x_3=a, && w_3=d_1x_1+d_2x_2+d_5x_5=d.\end{aligned}$$

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* T. A. Hirst: "On Cremonian Congruences," *Proc. London Math. Society*, Vol. XIV (1883), pp. 259–301. The treatment is synthetic, and no transformations are discussed.

If now a general line of the system be projected from (0, 0, 0, 0, 1) into the space $x_5 = 0$, its equations become

$$\alpha x_1 + \beta x_2 + \gamma x_3 = 0, \qquad d_5 \gamma (\alpha x_4 + a\gamma) = b\alpha\beta + \beta^2 c + \beta\gamma d.$$

By eliminating α the equation may be written

$$(b x_2 - c x_1)\beta^2 - (d_5 x_2 x_4 - b x_3 + d x_1)\beta\gamma + (a x_1 - x_3 x_4) d_5 \gamma^2 = 0, \qquad (2)$$

showing that the congruence can be arranged in a family of quadrics of index 2. The equation of the focal surface may be written in the form

$$\phi_4 = (d_5 x_2 x_4 + b x_3)^2 + x_1 [2 d d_5 x_2 x_4 + d^2 x_1 - 2 d b x_3 - 4 c x_3 x_4 - 4 a b d_5 x_2] = 0.$$

The two systems of generators on the quadric (2) are rationally separable, and may be written

$$\frac{b\beta - d_5 x_4 \gamma}{x_1} = \frac{c\beta^2 + d\beta\gamma - a d_5 \gamma^2}{\beta x_2 + \gamma x_3} = \sigma,$$

$$\frac{b\beta - d_5 x_4 \gamma}{c\beta^2 + d\beta\gamma - a d_5 \gamma^2} = \frac{x_1}{\beta x_2 + \gamma x_3} = \tau.$$

Since the congruences are each of class 6, and the focal surface has a double plane $x_1 = 0$, they are of symbol $(2, 6)_{II}$. The double points of the focal surface consist of the eight points common to the quadrics

$$b x_2 - c x_1 = 0, \quad d_5 x_2 x_4 - b x_3 + d x_1 = 0, \quad a x_1 - x_3 x_4 = 0,$$

and the vertices of the four quadric cones of the family. Of the former, two have an exceptional function: one is the vertex of a quintic cone K_5 having a double edge passing through six of the basis points, and the other is the vertex of a plane pencil K_1 lying in the singular plane. The other six basis points are vertices of cubic cones. The vertices of the quadric cones contained in the family of quadrics are all in the singular plane $x_1 = 0$.

In σ, τ the two cones K_5, K_1 are interchanged, each has a cubic cone at the other six basis points, but the cones are not the same for both. The quadric cones are common to σ and τ. The equations of these cones can be determined by finding the envelope of the plane passing through the singular point. One equation of the line of the congruence is illusory, and the parameters in the other will satisfy a relation with constant coefficients.

7. When the operation (Σ) is performed upon the focal surface, all the double points are principal elements. K_1 will go into the conic of contact lying in $x_1 = 0$, each of the K_3 vertices goes into a space quartic having a double point at

the vertex, the K_8 each into a sextic having a triple point and K_5 into a curve of order 10, with a five-fold point at the vertex, and passing through the basis points. All these curves are rational.

8. Given any plane section π of ϕ. Consider the ruled surface formed by all the lines of σ which touch ϕ along the quartic curve of section. Its order will be 4 plus the number of generators lying in π. Since the class of σ is 6, there are six bitangents lying in π, each counting as a double generator. Thus the order of the ruled surface is 16. Since its genus is 3, the residual double curve is of order 96, no part of which can consist of further double generators. The plane π cuts the curve c_{2i}, image of K_i, in $2i$ points, hence $2i$ generators of our ruled surface pass through K_i. The surfaces ϕ_4 and R_{16} intersect in a curve of order 64, consisting of c_4 in π counted twice, and a residual c_{28}, also counted twice. The latter will have $2i$ branches passing through every K_i. The double curve D_{96} on R_{16} meets ϕ_4 in all the vertices K_i and nowhere else. The plane π will cut each cone K_i in a plane c_i which touches c_4 in $2i$ points. The lines joining these points to the vertex K_i will therefore be torsal generators, the pinch-point of each being at the vertex. The total intersection of D_{96} and ϕ_4 is therefore of the form

$$2\left\{\frac{10.9}{2}+5+6\left(\frac{6.5}{2}+3\right)+4\left(\frac{4.3}{2}+2\right)+1+1\right\}.$$

9. If between the equations of a line of the congruence the parameter β be eliminated, a family of ruled cubics of index 2 results,

$$\alpha^2\left[b x_1 x_2 - c x_1^2\right] + \alpha\gamma\left[d_5 x_2^2 x_4 + b x_2 x_3 - 2c x_1 x_3 + d x_1 x_2\right] \\ + \gamma^2\left[d_5 \alpha x_2^2 - c x_3^2 + d x_2 x_3\right] = 0,$$

which has ϕ for its envelope, with the extraneous factor x_2^2. The double directrices are defined by

$$x_2 = 0, \quad \alpha x_1 + \beta x_3 = 0.$$

In this way we can obtain the properties first found by Stahl.*

10. The equations

$$\frac{a\lambda + b\mu}{c\lambda + d\mu} = \frac{x\lambda + y\mu}{z\lambda + w\mu} = \sigma$$

and the corresponding ones for τ also define congruences of order 2 and class 6, but the focal surface has no double plane; hence the congruence is of the first kind $(2, 6)_I$. The configuration of singular points and fundamental curves can be found as before. The essential difference between the two forms is that in $(2, 6)_{II}$ the vertices of the quadric cones belonging to the family of quadrics lie in a plane, while in $(2, 6)_I$ they do not. The present form can not be obtained from the cubic variety Γ of S_4.

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* W. Stahl: "Ueber Strahlensysteme zweiter Ordnung," *Crelle's Journal*, Vol. XCV (1883), pp. 297–316.

11. If in the equation Γ the space v_3 is of the form $a_4 x_4 + a_5 x_5$, the entire plane $x_4 = 0$, x_5 lies on Γ. There is no restriction in putting $a = 0$, since the equation of any space through this plane is of the form $k x_4 + l x_5 = 0$. Each space of this pencil will cut Γ in a quadric whose generators belong to conjugate systems. Four double points of Γ lie in the plane, and are basis points of the pencil of conics cut from the quadrics by the plane $x_4 = 0$, $x_5 = 0$.

On eliminating α as before, the system

$$\frac{b\beta - d_5 x_4 \gamma}{x_1 \beta} = \frac{c\beta + d\gamma}{x_2 \beta + x_3 \gamma} = \sigma$$

is seen to be a particular case of both the preceding forms. The planes $x_1 = 0$, $x_3 = 0$ and $x_4 = 0$ are double planes of the focal surface.

From the equation of the system of quadrics

$$\beta^2 (b x_2 - c x_1) + \beta\gamma (b x_3 - d_5 x_3 x_4 - d x_1) - \gamma^2 d_5 x_3 x_4 = 0$$

it is seen that four of the eight basis points lie in the plane $x_3 = 0$, and the other four lie in $x_4 = 0$. Besides the four cones whose vertices are in $x_1 = 0$, the congruence has one other singular point. This is the general congruence (2, 5).

12. If $b = 0$, the general congruence (2, 4) results. The singular planes of the focal surface are $x_1 = 0$, $x_3 = 0$, $x_4 = 0$, $\sigma = 0$.

Finally, the (2, 3) and (2, 2) congruences can be obtained from the equations given in a preceding paper.*

In the case of the general (2, 2) congruence, the same focal surface exists for six families of congruences. The operations $(\Sigma)(\mathrm{T})$ are now all commutative, and generate a group of order 32. In all the other cases enough undefined constants exist to make $(\Sigma \mathrm{T})$ of infinite order, but for certain values of these constants they may be of finite order.

Every congruence of order 2 *not having focal curves and of class less than* 7 *can be arranged on a family of quadric surfaces of index* 2, *whose envelope is the focal surface of order* 4. *For unrestricted coefficients these surfaces all have an infinite group of birational transformations except that of class* 4.

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* Snyder: "Surfaces Derived from the Cubic Variety Having Nine Double Points in Four-Dimensional Space," *Trans. Am. Math. Soc.*, Vol. X (1909), pp. 71–78.

Congruences of Order 3.

13. If the variety Γ which has no plane lying upon it be projected from a general point of S_4, the resulting congruences of order 3 can not in general be arranged on quadrics. The two conjugate congruences of order 3 and class 6 have the same focal surface of order 6.

If, however, $a=0$, and the system be projected from (00101) into $x_5=0$, an interesting case results. It will be more convenient to use a non-homogeneous parameter k. The projection of the first system gives

$$\frac{d_5 x_4 + k d}{x_4 + k x_3} = \frac{k(b-kc)+b_3 x_4}{k(x_1 - k x_2)} = \sigma \qquad C_3,$$

from which

$$(c x_3 - d x_2) k^3 + (d x_1 - d_5 x_2 x_4 + c x_4 - b x_3) k^2 \\ + (d_5 x_1 x_4 - b x_4 - b_3 x_3 x_4) k - b_3 x_4^2 = 0.$$

In the k-discriminant, x_4^2 is a factor.

Of the six quadric cones in this family of quadrics, three have their vertices in $x_4=0$, three in $d_5 x_3 - d = 0$. The point $(d_2, -d_1, 0, 0)$ is the vertex of a quartic cone having three cuspidal edges, belonging to C_3.

A second system is

$$\frac{k d + d_5 x_4}{k(b-kc)+b_3 x_4} = \frac{k x_3 + x_4}{k(x_1 - k x_2)} = \tau \qquad C_1.$$

This one has the same three quadric cones with vertices in the plane $x_4=0$, cubic cones with vertices at the singular points in the plane $d_5 x_3 - d = 0$, and a plane pencil in this plane with vertex at $(d_2, -d_1, 0, 0)$.

Finally, a third system is of the form

$$\frac{d - k x_3}{(d_5 - k)(k x_2 - c)} = \frac{x_4 (d_5 - k)}{b_3 (d - k x_3) + (d_5 - k)(k x_1 - b)} = \rho \qquad C_2,$$

which has a pencil in the plane $x_4=0$. It is like C_1, if the planes x_4, $d_5 x_3 - d = 0$ be interchanged. The other system of generators on the quadrics of C_2 is exactly C_3. Each of the operations (C_i) is of order 2, and the product of any two is of infinite order.

To obtain the equation of the cone of rays passing through the point $(d_2, -d_1, 0, 0)$ and belonging to C_1, substitute the coordinates of this point in C_1 and solve for τ, k. The first gives $\tau=0$; hence $kd + d_5 x_4 = 0$, $k x_3 + x_4 = 0$. The k-eliminant is $(d_5 x_3 - d) x_4 = 0$. The factor x_4 corresponds to $k=0$; hence the pencil $d_5 x_3 - d = 0$, $(d_2, -d_1, 0, 0)$ belongs to C_1.

Similarly, in C_2 we have $\rho = 0$, $x_4 = 0$ for the same point, and for C_3 the coordinates of the point satisfy one equation identically. The other gives $\sigma(d_2 + k d_1) = \bar{b}$, $\bar{b}$ being $d_2 b_1 - d_1 b_2$. On substituting this value of σ in the second equation we have the equation of a plane always passing through the singular point. The envelope of this plane is a quartic cone having three cuspidal edges. In the same way the cones belonging to the other singular points are determined.

The duals of all the congruences of order 2 discussed before can also be expressed in this way.

14. Another case of considerable interest was mentioned by Roccella.* Let $(0_i, \pi_i)$ $(i = 1, 2, 3)$ represent three projective plane pencils. Lines joining corresponding lines will generate a quadric, and the quadrics thus defined will compose a congruence G. The directrices of the quadrics define another congruence D. The planes π_i, π_k will intersect in a line $(\pi_i \pi_k)$ which is cut by the two pencils in projective ranges. Let P_{ik}, P'_{ik} be the self-corresponding points, and d_i, d_k; d'_i, d'_k the pairs of lines of the two pencils passing through them. Let δ_m be the line of $(0_m, \pi_m)$ corresponding to d_i. The lines of the pencils (P_{ik}, δ_m), (P'_{ik}, δ'_m) obviously belong to G. Let the plane (d_i, d_k) cut δ_m in D_m. The pencil $(D_m, d_i d_k)$ will also belong to G. The lines of the pencils $(P_{ik}, d_i d_k)$ and $(D_m (P_{ik}, \delta_m))$ likewise belong to D.

The quadric cones belonging to the family of quadrics therefore break up into planes. The congruences G, D have twelve singular points, at each of which the singular lines form a plane pencil.

The pencils $(0_i, \pi_i)$ belong to D, but not to G. If l_i, l_k, l_m form any triad of corresponding lines, every line through 0_i cutting l_k, l_m will evidently belong to D. The lines which meet l_k, l_m describe a quadratic complex. The complex cone having 0_i as vertex will therefore belong to G. Similarly, all the lines of π_i belonging to this complex will belong to G. From their method of generation both congruences are self-dual; hence:

The congruence generated by the lines which meet corresponding lines of three projective pencils is of order and class 3; it contains fifteen singular points and fifteen singular planes. At twelve of these points the singular lines form plane pencils, and at the others, quadric cones. The three remaining planes contain conics passing

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* "Sagli enti geometrici dello spazio di rette generati dalle intersezioni de' complessi corrispondenti in due o più fasci proiettivi di complessi lineari." Piazza Armerina, 1882. See Art. 10.

through the vertices of the cones. The conjugate congruence is of order and class 3, and contains fifteen singular points, at each of which the singular lines form a plane pencil.

15. Analytically we may write

$$\begin{aligned} x_1 + \lambda x_2 = 0, \quad x_3 = 0 \quad &\text{for} \quad (0_1,\ \pi_1), \\ \lambda x_1 + x_2 = 0, \quad x_4 = 0 \quad &\text{for} \quad (0_2,\ \pi_2), \\ a_2 x_2 + a_3 x_3 + a_4 x_4 = 0, \quad x_1 + x_3 + x_4 = 0 \quad &\text{for} \quad (0_3,\ \pi_3), \end{aligned}$$

wherein $a_i = b_i + \lambda c_i$, and b_i, c_i are constants.

The equations of a line of G now have the form

$$\frac{(\lambda^2 - 1)(a_3 - a_4)x_3 - (a_2\lambda + a_4)(x_1 + \lambda x_2)}{a_2(x_1 + \lambda x_2 + x_3)} = \frac{\lambda x_1 + x_2}{x_4} = \sigma.$$

The focal surface is the same for G and D. It is of order and class 8 and has a cuspidal curve of order 12. In general the cycle (GD) is of infinite order. The surface is the self-dual surface of Kummer and Cayley.*

CORNELL UNIVERSITY, *July*, 1909.

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* E. E. Kummer: "Ueber diejenigen Flächen, welche mit ihren reciprok polaren Flächen von derselben Ordnung sind und die gleichen Singularitäten besitzen," *Berliner Monatsberichte*, 1878, pp. 25–37; Cayley: "On Sibi-Reciprocal Surfaces," same, pp. 309–313.

The Apparent Size of a Closed Curve.

BY ARTHUR C. LUNN.

1. The apparent size of a closed curve in space, or the solid angle subtended by it as seen from any point, is defined as the solid angle of a cone with vertex at the point of sight and with the curve as directrix. This is measured, according to the definition of Gauss, by the area cut by the cone from a sphere of unit radius with center at the given point. It is equivalent to the apparent size of any surface bounded by the curve, provided a suitable convention be adhered to as to the algebraic sign of each element, according to whether one side or the other is visible from the point. Its value is then given by the familiar integral

$$\omega = -\int \frac{\cos(n, r)\, dS}{r^2} = -\int \frac{x\cos(n, x) + y\cos(n, y) + z\cos(n, z)}{r^3}\, dS \qquad (1)$$

extended over a surface bounded by the curve, with positive side at each point fixed by the direction of the normal n. Here the point of sight is taken as origin of coordinates and an element of solid angle is considered positive when the origin is on the positive side of the corresponding element of surface, r being the radius vector from the origin to a point of the surface. The explicit computation of ω is required, for example, in the determination of the magnetic potential of a closed linear electric current.*

The integral in (1) may be evaluated as the sum of three double integrals, extended over the areas bounded by the projections of the curve on the coordinate planes, or as one double integral extended over a plane area into which the surface is mapped by means of a representation of the coördinates of its points in terms of two independent parameters. But inasmuch as the surface here is purely auxiliary and is arbitrary except in that it is to be bounded by the given curve, it seems more natural to seek a method of computation such as to use the curve itself as realm of integration, for instance by integration with respect to a parameter t in terms of which the coordinates of the points of the curve may be expressed.

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* Gauss: "Allgemeine Theorie des Erdmagnetismus," *Gesammelte Werke*, V, p. 170.

One such method is given by the ordinary formula

$$\omega = \int (1 - \cos\phi)\, d\theta = \int (1 - \cos\phi)\, \theta' \, dt, \tag{2}$$

where θ, ϕ are the longitude and colatitude of the point where the unit sphere is intersected by the vector $\mathbf{r}$ from the origin to a point of the curve. Here the use of t as a variable of integration, in terms of which the coordinates may be considered as single-valued, relieves the ambiguity as to the interval of variation of θ arising from the fact that the polar axis, being arbitrary, may or may not be linked with the curve. For comparison with other formulas to be considered it will be convenient to throw this into a form containing rectangular coordinates, which will here be done by means of a vector formula independent of any particular coordinate system.

Let $\mathbf{a}$ be a constant vector, of length a, in the direction of the polar axis implied in equation (2), and let $\mathbf{r}_1$, $\mathbf{a}_1$ be unit vectors in the directions of $\mathbf{r}$ and $\mathbf{a}$ respectively, so that $\mathbf{r} = r\mathbf{r}_1$, $\mathbf{a} = a\mathbf{a}_1$. Then if t be thought of as time, θ' is the angular velocity of the plane of the vectors $\mathbf{r}$, $\mathbf{a}$ about the fixed vector $\mathbf{a}$ as axis; and $\cos\phi = \mathbf{r}_1 . \mathbf{a}_1$.* Now a vector normal to the plane in question is $\mathbf{n} = n\mathbf{n}_1 = \mathbf{r} \times \mathbf{a}$, so that the derivative $\mathbf{n}_1'$ of the unit normal vector is numerically equal to θ'. From this come:

$$\mathbf{n}' = n\mathbf{n}_1' + n'\mathbf{n}_1 = \mathbf{r}' \times \mathbf{a}, \quad n^2 = (\mathbf{r} \times \mathbf{a})^2, \quad nn' = (\mathbf{r}' \times \mathbf{a}) . (\mathbf{r} \times \mathbf{a}),$$
$$n^2\mathbf{n}_1' = n(\mathbf{r}' \times \mathbf{a}) - \mathbf{n}_1(\mathbf{r}' \times \mathbf{a}) . (\mathbf{r} \times \mathbf{a}),$$

and from the last, by squaring and reducing:

$$n^4 \mathbf{n}_1'^2 = \{(\mathbf{r} \times \mathbf{a}) \times (\mathbf{r}' \times \mathbf{a})\}^2 = \{[\mathbf{r}'\mathbf{r}\mathbf{a}]\,\mathbf{a}\}^2, \tag{3}$$

which shows that the vector of angular velocity of the plane may be taken as

$$\mathbf{u} = \frac{[\mathbf{r}'\mathbf{r}\mathbf{a}]}{(\mathbf{r} \times \mathbf{a})^2}(-\mathbf{a}), \tag{4}$$

which is numerically equal to $\mathbf{n}_1'$, but is parallel to the axis $\mathbf{a}$ while $\mathbf{n}_1'$ is perpendicular. This formula may be obtained also directly by a geometric construction, with $\mathbf{r}'$ treated as the velocity of a point moving along the curve, so that $-[\mathbf{r}'\mathbf{r}\mathbf{a}]/a$ is the moment of the velocity with respect to the axis. The sense of $\mathbf{u}$, left undetermined by (3), is here so chosen that when $\mathbf{r}'$, $\mathbf{r}$, $\mathbf{a}$ form a right-handed triple of vectors the direction of $\mathbf{u}$ shall be opposite to that of $\mathbf{a}$. Thus if the forward circuit of the curve appear counter-clockwise as seen from what is taken in (1) as the positive side of the surface, the resulting value of the solid angle is

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* The vector notation is throughout that of Gibbs-Wilson: "Vector Analysis."

$$\omega = \int \left(1 - \frac{\mathbf{r}\cdot\mathbf{a}}{r a}\right) \frac{[\mathbf{r}'\mathbf{r}\mathbf{a}]\, a}{(\mathbf{r} \times \mathbf{a})^2}\, dt, \tag{5}$$

which in Cartesian rectangular coordinates is

$$\left.\begin{aligned} \omega &= \int \left(1 - \frac{a_1 x + a_2 y + a_3 z}{a r}\right) \frac{a}{n^2} \begin{vmatrix} x' & y' & z' \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix} dt, \\ r &= \sqrt{x^2 + y^2 + z^2}, \qquad a = \sqrt{a_1^2 + a_2^2 + a_3^2}, \\ n^2 &= (a_2 z - a_3 y)^2 + (a_3 x - a_1 z)^2 + (a_1 y - a_2 x)^2 \\ &= a^2 r^2 - (a_1 x + a_2 y + a_3 z)^2, \end{aligned}\right\} \tag{6}$$

where a_1, a_2, a_3 are the scalar components of **a**. In this deduction the polar axis has any arbitrary direction with respect to the rectangular axes; if it be taken to coincide with the z-axis, and t be the arc of the curve, this formula reduces to one used by Maxwell.*

The integrand in this formula is independent of the particular set of axes used for the resolution of the vectors; and the integral is independent of the special choice of the parameter t, but it contains the direction of the arbitrary vector **a** and thus depends apparently on the ratios of the scalar constants a_1, a_2, a_3. The geometric source of the formula shows, however, that the integral around a closed curve must be independent of those constants, for any continuous variation of them which does not introduce singularities into the integrand.

If the curve intersect the axial line, then at such a point **r** becomes parallel to **a** and the divisor n^2 vanishes. Since **a** is arbitrary, this could of course be avoided for any particular curve, but there is a feature of theoretical interest in such a case.

If the intersection is on the positive half of the axis of **a**, so that $\phi = 0$, the vanishing of other factors keeps the integrand finite; but if it is on the negative half, with $\phi = \pi$, the integrand becomes infinite. In the latter case a suitable value of ω might be sought by a limiting process, for which purpose a short arc of the curve in the neighborhood of the intersection is replaced by a circular arc, for instance, with center on the axis. But the result then depends upon which way this arc lies with respect to the axis, there being in fact, for arcs on opposite sides of the axis, two possible limiting values of the integral differing by 4π, which is the limiting value of the integral in (2) or (5) for a small circle with center on the negative part of the axis. Thus a variation of the vector **a** such as to make its negative continuation pass across the curve produces an abrupt change of 4π in the value of the integral. The obvious interpretation is that this 4π is the apparent size of the whole "sky," or the cyclic constant which occurs additively an arbitrary integral number of times in the general many-valued determination of ω. Moreover, if two different expressions for the integrand in (6) be formed, corresponding to different choices of the vector **a**, their difference proves to be the exact derivative of a certain function of the coordinates which is expressible in terms of inverse trigonometric tangents and is many-valued with the same cyclic constant 4π.

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* "Electricity and Magnetism," Vol. II, Ed. 3, p. 39.

2. Another method of computation* depends on the relation $\omega = 2\pi - \sigma$, generalized from a familiar theorem relative to polar spherical triangles, in which σ is the length of that curve on the unit sphere which is polar to the intersection of sphere and cone. To translate this into an explicit general formula, let it be noticed that if the vector $\mathbf{r}$ goes to a point of the given curve then the unit vector of the vector $\mathbf{p} = \mathbf{r} \times \mathbf{r}'$ goes to the corresponding point of the polar curve on the sphere. Then

$$p\,\mathbf{p}_1' = \mathbf{r} \times \mathbf{r}'' - p'\,\mathbf{p}_1, \quad p^2 = (\mathbf{r} \times \mathbf{r}')^2, \quad p\,p' = (\mathbf{r} \times \mathbf{r}') \cdot (\mathbf{r} \times \mathbf{r}''),$$

and from the first of these, by squaring and reducing:

$$p^4\,\mathbf{p}_1'^2 = r^2\,[\mathbf{r}'', \mathbf{r}', \mathbf{r}]^2. \tag{7}$$

Here $\mathbf{p}_1'$ is numerically the derivative of the arc of the polar curve, so that with a conventional choice of sign in taking the square root in (7) the solid angle is

$$\omega = 2\pi - \int \frac{r\,[\mathbf{r}'', \mathbf{r}', \mathbf{r}]}{(\mathbf{r} \times \mathbf{r}')^2}\,dt, \tag{8}$$

or in Cartesian coordinates:

$$\left.\begin{aligned} \omega &= 2\pi - \int \frac{r}{p^2} \begin{vmatrix} x'' & y'' & z'' \\ x' & y' & z' \\ x & y & z \end{vmatrix} dt, \\ p^2 &= (y z' - z y')^2 + (z x' - x z')^2 + (x y' - y x')^2 \\ &= r^2 q^2 - (x x' + y y' + z z')^2, \\ q^2 &= x'^2 + y'^2 + z'^2. \end{aligned}\right\} \tag{9}$$

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* Used for example by Schwartz: *Gesammelte Werke*, I, p. 316.

To put this into a form using polar coordinates, for comparison with (2), let (x, y, z) be written $(r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi)$; then it reduces to

$$\omega = 2\pi - \int \frac{(\phi'' \theta' - \theta'' \phi') \sin\phi - 2\,\theta' \phi'^2 \cos\phi - \theta'^3 \sin^2\phi \cos\phi}{\phi'^2 + \theta'^2 \sin^2\phi}\, dt. \qquad (10)$$

The divisor p^2 vanishes at a point of the curve where the tangent line coincides with the radius vector. At such a point the integrand may remain finite if also the osculating plane pass through the origin; but in general it becomes infinite and the value of ω needs to be found by a limiting process, involving a system of curves obtained by warping the original so as to keep p different from zero. The typical case of such a critical point is that where the curve is such that its trace on the unit sphere presents a cusp. Then, according to whether the curve is deformed so as to change this cusp to a rounded arc or to some kind of loop, the limiting process gives a result indeterminate by an additive multiple of 4π, like that mentioned in § 1.

In contrast with the formula of § 1, the present one contains no arbitrary constants, but it involves the second derivatives of the coordinates and thus implies stronger conditions on the curve. For example, equation (5) or (6) could be used directly for a polygon or other curve having in general a definite tangent but with a suitably restricted set of points of discontinuity in the direction of that tangent; while in such a case equation (8) or (9) would be inapplicable unless for each such point there be added a corrective term measured by the corresponding change in direction of the line tangent to the trace on the unit sphere.

3. The method now to be developed consists in a transformation of the surface integral of equation (1) into a line integral by means of Stokes's theorem:

$$\int \operatorname{curl} \mathbf{v} \cdot d\mathbf{a} = \int \mathbf{v} \cdot d\mathbf{r}, \qquad (11)$$

where $d\mathbf{a}$ is the vector equivalent of the scalar element of area dS.

If $\mathbf{R}$ be put for the Newtonian vector $-\mathbf{r}_1/r^2$, then equation (1) may be written

$$\omega = \int \mathbf{R} \cdot d\mathbf{a}. \qquad (12)$$

Thus if there is a vector function $\mathbf{v}$ such that $\operatorname{curl} \mathbf{v} = \mathbf{R}$, then under the analytic conditions sufficient for Stokes's theorem the solid angle can be computed from

$$\omega = \int \mathbf{v} \cdot \mathbf{r}'\, dt. \qquad (13)$$

Since **R** is a solenoidal vector function, there must exist for it such a vector potential **v**, whose determination in terms of rectangular coordinates involves the integration of the system of differential equations:

$$\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}=-\frac{x}{r^3},\quad \frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}=-\frac{y}{r^3},\quad \frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}=-\frac{z}{r^3}, \tag{14}$$

which condition its scalar components (v_1, v_2, v_3).

By these equations alone **v** is not uniquely determined, but any two solutions differ by a potential vector ∇U or one whose components are partial derivatives of a single scalar function U. This U may be many-valued, but its discontinuities and those of ∇U must be allowed for by corresponding corrective terms in the application of Stokes's theorem. It may be predicted that the only indeterminateness thus introduced will be such that the values of ω computed by means of two different expressions for **v** will differ by a multiple of 4π. A general proof of this is not attempted here, but a verification is given below for some particular cases.*

A particular solution of equations (14), found by a known tentative process, is the following:

$$\left.\begin{aligned} v_1&=\frac{1}{3r}\left(\frac{y}{r+z}-\frac{z}{r+y}\right),\\ v_2&=\frac{1}{3r}\left(\frac{z}{r+x}-\frac{x}{r+z}\right),\\ v_3&=\frac{1}{3r}\left(\frac{x}{r+y}-\frac{y}{r+x}\right),\end{aligned}\right\} \tag{15}$$

which may be written

$$\mathbf{v}=\tfrac{1}{3}\mathbf{r}_1\times\mathbf{w};\quad \mathbf{w}=\left(\frac{1}{r+x},\ \frac{1}{r+y},\ \frac{1}{r+z}\right). \tag{16}$$

Then the general solution may be written $\mathbf{v}+\nabla U$, where the scalar function U is arbitrary. The special form of **v** just found gives, by substitution in (13):

$$\omega=\tfrac{1}{3}\int[\mathbf{r}',\mathbf{r}_1,\mathbf{w}]\,dt, \tag{17}$$

which in Cartesian coordinates is

$$\omega=\frac{1}{3}\int\begin{vmatrix} x' & y' & z'\\ x & y & z\\ \dfrac{1}{r+x} & \dfrac{1}{r+y} & \dfrac{1}{r+z}\end{vmatrix}dt. \tag{18}$$

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* For a theory of many-valued functions in a Riemann space see a paper by Dixon, *Proceedings of the London Mathematical Society*, Ser. 2, Vol. I, pp. 415-436.

With respect to this formula a qualification must be made, however, to cover cases where the vanishing of any of the quantities $(r+x,\ r+y,\ r+z)$ makes $\mathbf{v}$ infinite at any point involved in the integrals of equation (11). Since continuity of $\mathbf{v}$ and of curl $\mathbf{v}$ is presupposed in Stokes's theorem, a corrective term is needed for every point where the negative part of a coordinate axis pierces the surface implied in (11), even when at all points of the curve itself the integrand in (18) remains finite.

To find the magnitude of this correction it suffices to consider a special case, such as a circle with center on the negative z-axis and lying in a normal plane, represented by

$$x=\rho\cos t,\ y=\rho\sin t,\ z=-b,$$

where ρ, b are positive constants, so that r is the constant $\sqrt{\rho^2+b^2}$. For the integral in (18) this gives

$$\frac{1}{3r}\int_0^{2\pi}dt\left\{-\frac{\rho^2}{r-b}-\frac{b\rho\sin t}{r+\rho\sin t}-\frac{b\rho\cos t}{r+\rho\cos t}\right\}=2\pi\left(\frac{1}{3}-\frac{b}{r}\right),$$

while the true value for ω is $2\pi(1-b/r)$. Hence it may be inferred in general that for every point where the negative part of a coordinate axis, when taken in the direction from the origin to infinity, pierces the surface from its positive or negative side, a corresponding correction $4\pi/3$, positive or negative respectively, must be added to the integral in (18). Thus the passage of the curve across a negative semi-axis during continuous deformation would be marked by a discontinuity of $4\pi/3$ in the value of the integral.

For example, suppose ω_0 is the value of the integral for a curve linked with k of the negative semi-axes. By reversal of the sense of integration the same curve may be considered to define also the complementary solid angle with $3-k$ corrective terms. These mutually complementary solid angles are then $\omega_0+4\pi k/3$ and $-\omega_0+4\pi(3-k)/3$, whose sum is 4π, as it should be.

If two curves lie on the same cone, with vertex at the point of sight, and are represented in terms of the same parameter t, by a correspondence of those points which lie on the same linear element of the cone, then if the vector to one curve be $\mathbf{r}$, the corresponding vector to the other curve is $\phi\mathbf{r}$, where ϕ is a scalar function of t. On account of homogeneity of degree zero in the coordinates, the integrand in (17) is therefore the same for both curves, since the derivative ϕ' has as coefficient a vanishing determinant. A similar remark applies to the alternative formulas (5) and (8).

On Linear Transformations which Leave an Hermitian Form Invariant.

By J. I. Hutchinson.

Any Hermitian form whose determinant is not zero may be reduced by a linear transformation to

$$H = \sum_{i=1}^{n} \lambda_i z_i \bar{z}_i, \tag{1}$$

in which the coefficients λ_i are real and p of them are positive. The number p is invariant for any given form.* This fact will be referred to as the *Law of Inertia.*

It is the object of the present paper to consider some of the properties of linear transformations for which H is invariant.† For this purpose it is sufficient to consider the cases in which p does not exceed $\frac{1}{2}n$, since the forms H and $-H$ are invariant for the same substitutions.

§ 1. *The Transformations S and their Classification.*

Let the equations

$$(S) \qquad z'_k = \sum_{i=1}^{n} c_{ki} z_i \qquad (k = 1, 2, \ldots, n)$$

define any linear transformation whose determinant is not zero. If H is unaltered by S, the coefficients c_{ki} must satisfy the conditions

$$\sum_{k=1}^{n} \lambda_k c_{ki} \bar{c}_{kj} = \begin{cases} \lambda_i, & \text{when } j = i, \\ 0, & \text{“} \quad j \neq i. \end{cases} \tag{2}$$

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* See A. Loewy: “Ueber bilineare Formen mit conjugirt imaginären Variablen,” *Mathematische Annalen*, Vol. 50 (1898), p. 563.

† We follow the methods used by Picard for the case $n = 3$ in his memoir, “Sur les formes quadratiques ternaires indéfinies à indeterminées conjuguées et sur les fonctions hyperfuchsiennes correspondantes,” *Acta Mathematica*, Vol. V (1884), § 16, rather than the symbolic methods of Frobenius as employed by Loewy, *loc. cit.*

If the n equations (S) be multiplied by $\lambda_k \bar{c}_{kj}$ $(k = 1, 2, \ldots, n)$ and added, we obtain, after using (2), the inverse transformation S^{-1} in the form

$$(S^{-1}) \qquad z_i = \frac{1}{\lambda_i} \sum_{k=1}^{n} \lambda_k \bar{c}_{ki} z'_k \qquad (i = 1, 2, \ldots, n).$$

Since S^{-1} also leaves H invariant its coefficients satisfy the conditions

$$\sum_{i=1}^{n} \frac{1}{\lambda_i} c_{li} \bar{c}_{ki} = \begin{cases} \lambda_k^{-1}, & \text{when } l = k, \\ 0, & \text{,, } \quad l \neq k, \end{cases} \tag{2'}$$

which are equivalent to (2).

In what follows we shall not distinguish between two substitutions whose corresponding coefficients differ by a common factor. Such a factor is necessarily of the form $e^{i\phi}$.

The transformation S contains $2n^2$ real constants which are subjected to n^2 real conditions (2). Accordingly, *the substitution S contains n^2 arbitrary real parameters.**

For function-theory purposes it is often desirable to introduce non-homogeneous variables, which will be defined by the equations

$$u_\alpha = z_\alpha / z_n \qquad (\alpha = 1, 2, \ldots, n-1). \tag{3}$$

Let u_α be separated into real and imaginary parts, $u_\alpha = u'_\alpha + i u''_\alpha$. The values of these $2(n-1)$ variables will be associated with the points of a flat space R of $2(n-1)$ dimensions.

Denote by F the function $H / z_n \bar{z}_n$ of $u_1, \bar{u}_1, \ldots, u_{n-1}, \bar{u}_{n-1}$. The locus

$$F = 0 \tag{4}$$

is transformed into itself by the substitution S, while any point not on (4) is transformed into a point on the same side of this locus unless $p = \frac{1}{2} n$. We will call the region defined by the inequality $F > 0$ the inside of F.

§ 2. *The Fixed Points.*

We now proceed to the consideration of the points of R which remain fixed for the transformation S. For convenience we will return to the homogeneous notation. At a fixed point we have

$$z'_k = \rho z_k \qquad (k = 1, 2, \ldots, n).$$

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* The transformation may be put in a form in which the superfluous parameters are eliminated from the coefficients. See the foot-note to page 200. From the definition of T there given it is readily found to contain n^2 arbitrary real parameters.

Substituting in the equations (S) we obtain the conditions

$$c_{k1} z_1 + c_{k2} z_2 + \dots + (c_{kk} - \rho) z_k + \dots + c_{kn} z_n = 0 \quad (k = 1, 2, \dots, n). \tag{5}$$

If C denote the determinant of these equations, then, as is well known, the value of ρ is found from the characteristic equation $C = 0$.

If, on the other hand, we substitute $z' = \rho z$ in the equations (S^{-1}) and divide each one by ρ, we obtain

$$\lambda_1 \bar{c}_{1i} z_1 + \lambda_2 \bar{c}_{2i} z_2 + \dots + \lambda_i (\bar{c}_{ii} - \rho^{-1}) z_i + \dots + \lambda_n \bar{c}_{ni} z_n = 0 \quad (i = 1, 2, \dots, n).$$

Denoting the determinant of these equations by C', we obtain for ρ the equation $C' = 0$. After dividing out $\lambda_1 \lambda_2 \dots \lambda_n$, we observe that if in the resulting equation every letter is replaced by its conjugate, the equation so obtained is the same as $C = 0$ except that ρ is replaced by $1/\bar{\rho}$. Hence, *if* ρ *is a root of* $C = 0$, *then* $1/\bar{\rho}$ *is also a root of the same equation.*[*]

We therefore have a number q of pairs of roots ρ_1, $\rho_2 = 1/\bar{\rho}_1$, ρ_3, $\rho_4 = 1/\bar{\rho}_3$, $\dots$, ρ_{2q-1}, $\rho_{2q} = 1/\bar{\rho}_{2q-1}$ whose absolute values are reciprocals, and different from unity. The remaining roots $\rho_{2q+1}, \dots, \rho_n$ are then unimodular. Suppose these n roots are all distinct. By a linear transformation T the substitution S may be put in the form $\zeta'_k = \rho_k \zeta_k$ $(k = 1, 2, \dots, n)$. Expressing H in terms of the variables ζ by means of T it takes the form

$$H = \sum_{i=1}^{q} (a_i \zeta_{2i-1} \bar{\zeta}_{2i} + \bar{a}_i \zeta_{2i} \bar{\zeta}_{2i-1}) + \sum_{\gamma=2q+1}^{n} a_\gamma \zeta_\gamma \bar{\zeta}_\gamma.$$

Now make the transformation

$$\zeta_{2i-1} = (\eta_{2i-1} + \eta_{2i})/a_i, \quad \zeta_{2i} = \tfrac{1}{2}(\eta_{2i-1} - \eta_{2i}), \quad \zeta_\gamma = \eta_\gamma,$$

and we obtain

$$H = \sum_{i=1}^{q} (\eta_{2i-1} \bar{\eta}_{2i-1} - \eta_{2i} \bar{\eta}_{2i}) + \sum_{\gamma=2q+1}^{n} a_\gamma \eta_\gamma \bar{\eta}_\gamma. \tag{6}$$

According to the Law of Inertia q can not exceed p. It follows that when the roots of $C = 0$ are all distinct the substitution S belongs to one of $p + 1$ different types according to the value of q. When $q = 0$, S will be called *elliptic.* In other cases it will be called *hyperbolic of the q-th type* ($q = 1, 2, \dots, p$).

Our next problem is to determine the positions of the fixed points of S. Suppose at first that the roots of $C = 0$ are all distinct. In that case we have

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[*] This result is obtained by Loewy (*loc. cit.*, p. 566) by an application of the symbolic methods and results of Frobenius to the Hermitian form. For the case $n = 3$ it was first given by Picard, *Acta Mathematica*, Vol. V (1884), p. 164.

n distinct fixed points $P_1, P_2, \ldots, P_n$ corresponding respectively to the values $\rho_1, \rho_2, \ldots, \rho_n$ of ρ. At the fixed points P_k the equations (S) reduce to $z_i' = \rho_k z_i \, (i = 1, 2, \ldots, n)$. If H' denote the transformed form $\Sigma \lambda_i z_i' \bar{z}_i'$, then at P_k we have the relation $H' = \rho_k \bar{\rho}_k H = H$, whence $(\rho_k \bar{\rho}_k - 1) H = 0$. If $|\rho_k|$ is not 1, it follows that H is 0. Hence, *$2q$ fixed points of a hyperbolic substitution of the q-th type are on the locus $F = 0$.*

When the roots ρ_k are all distinct, H may be reduced to the form (6) and therefore the coordinates of the fixed point $P_\gamma \, (\gamma = 2q + 1, \ldots, n)$ are $\eta_k = 0 \, (k \neq \gamma)$. If these be substituted in H, it reduces to $a_\gamma \eta_\gamma \bar{\eta}_\gamma$ and will therefore be positive or negative according to the sign of a_γ. On account of the invariance of the number p and since (6) has already q positive terms, it follows that *$p - q$ of the $n - 2q$ remaining fixed points P_γ are inside $F = 0$ and $n - p - q$ are outside.* Applying this result to the case $q = 0$ we may say that *p fixed points of an elliptic substitution are inside, and $n - p$ are outside the invariant locus F.*

The case of fixed points when the characteristic equation has equal roots will be considered later.

§ 3. *Reduction of H to the Canonical Form. General Case.*

Before taking up the case of equal roots ρ_k, we will first give some general transformations relative to the reduction of any Hermitian form to a canonical form. Let H be written in the most general form $\Sigma a_{ik} \zeta_i \bar{\zeta}_k$, assuming the determinant A of the coefficients to be different from zero. If for a particular coefficient a_{mm} of the main diagonal of A the corresponding minor A_{mm} is not zero, then a first step in the reduction of H to the canonical form is made by means of the transformation

$$\left.\begin{aligned} \zeta_\kappa &= \eta_\kappa + (A_{\kappa m}/A_{mm})\, \eta_m \qquad (\kappa \neq m,\ \kappa = 1, \ldots, n) \\ \zeta_m &= \eta_m . \end{aligned}\right\} \quad \text{(I)}$$

Expressed in terms of the new variables H becomes $H_1 + (A/A_{mm})\, \eta_m \bar{\eta}_m$, in which $H_1 = \sum\limits_{\kappa, \lambda} a_{\kappa\lambda} \eta_\kappa \bar{\eta}_\lambda$, the indices κ, λ taking all integral values from 1 to n with the exception of m. The determinant of the form H_1 is A_{mm}, which is not zero by hypothesis. If any minor of A_{mm} corresponding to an element of its main diagonal is not zero, we make another substitution of type I, and thus continue as long as possible. If a stage is reached at which (I) can no longer be used, we employ a transformation of the following type.

Suppose that in A the minor A_{mm} is zero but that the minor A_{1m} is not zero. In that case we can apply the transformation*

$$\left.\begin{aligned} \zeta_\lambda &= \eta_\lambda + (A_{\lambda m}/A_{1m})\,\zeta_1, \\ (A/A_{1m})\,\zeta_1 &= \tfrac{1}{2}\left[(1-a_{mm})\,\eta_1 + (1+a_{mm})\,\eta_m\right] - \sum_\lambda a_{\lambda m}\,\eta_\lambda, \\ \zeta_m &= \eta_1 - \eta_m, \end{aligned}\right\} \quad \text{(II)}$$

in which λ takes all values from 2 to n with the exception of m. The form H reduces to

$$\eta_1\bar{\eta}_1 - \eta_m\bar{\eta}_m + \sum_{\lambda,\mu} a_{\lambda\mu}\,\eta_\lambda\,\bar{\eta}_\mu \qquad (\lambda,\ \mu = 2,\ 3,\ \ldots,\ m-1,\ m+1,\ \ldots,\ n).$$

For, from the first equation (II) we obtain by changing to conjugate imaginary values, multiplying by $a_{i\lambda}$, and summing with respect to λ,

$$a_{i1}\,\bar{\zeta}_1 + \sum_\lambda a_{i\lambda}\,\bar{\zeta}_\lambda = a_{i1}\,\bar{\zeta}_1 + \sum_\lambda a_{i\lambda}\left[\bar{\eta}_\lambda + (\bar{A}_{\lambda m}/\bar{A}_{1m})\,\bar{\zeta}_1\right]. \tag{7}$$

Since $a_{i\lambda} = \bar{a}_{\lambda i}$, $\bar{A} = A$, and $A_{mm} = 0$, the terms in the right member which contain $\bar{\zeta}_1$ vanish unless $i = m$, in which case they reduce to $(A/\bar{A}_{1m})\,\bar{\zeta}_1$ and we have

$$a_{m1}\,\bar{\zeta}_1 + \sum_\lambda a_{m\lambda}\,\bar{\zeta}_\lambda = \sum_\lambda a_{m\lambda}\,\bar{\eta}_\lambda + (A/\bar{A}_{1m})\,\bar{\zeta}_1. \tag{7'}$$

Multiply (7) by ζ_i and sum with respect to i for all values except m. This gives

$$\sum_{\alpha,\beta} a_{\alpha\beta}\,\zeta_\alpha\,\bar{\zeta}_\beta = \sum_{\lambda,\mu} a_{\lambda\mu}\,\eta_\lambda\,\bar{\eta}_\mu, \tag{8}$$

in which α, β take all values except m, and λ, μ all values except 1 and m. Now write H in the form

$$\sum a_{\alpha\beta}\,\zeta_\alpha\,\bar{\zeta}_\beta + \sum a_{\alpha m}\,\zeta_\alpha\,\bar{\zeta}_m + \sum a_{m\alpha}\,\zeta_m\,\bar{\zeta}_\alpha + a_{mm}\,\zeta_m\,\bar{\zeta}_m$$

and use the second and third equations (II) together with the relations (7′) and (8). This leads to the result previously indicated.

§ 4. *Reduction of H when C Has Equal Roots.*

Suppose that the characteristic equation $C = 0$ has a number of equal roots $\rho_1 = \rho_2 = \ldots = \rho_m = \rho$. There are two cases to be considered, according as ρ is unimodular or not.

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* This, and (I), are adapted from Gundelfinger, "Ueber die Transformation einer quadratischen Form in eine Summe von Quadraten," *Crelle*, Vol. XCI (1881), p. 228.

In the first case we suppose S reduced to the normal form

$$\begin{vmatrix} M_1 & 0 & \dots & 0 \\ 0 & M_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & M_{\mu+2} \end{vmatrix} \tag{9}$$

in which the M_i are square matrices defined as follows:

$$M_i = \begin{vmatrix} \rho & \rho & 0 & \dots & 0 \\ 0 & \rho & \rho & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \rho & \rho \\ 0 & \dots & \dots & 0 & \rho \end{vmatrix} \quad (i = 1, \dots, \mu), \quad M_{\mu+1} = \begin{vmatrix} \rho & 0 & \dots & 0 \\ 0 & \rho & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \rho \end{vmatrix}.$$

It is not necessary to specify the form of $M_{\mu+2}$, except that it does not depend on ρ. Denote the orders of these matrices by $e_1, e_2, \dots, e_\mu, e_{\mu+1}$ respectively. The integers $e_1, \dots, e_\mu$ are the exponents of those elementary divisors of C containing the root ρ which are greater than unity. We suppose them arranged in decreasing order of magnitude, $e_1 \geqq e_2 \geqq e_3 \dots \geqq e_\mu \geqq 2$.

The condition that H be invariant for S may be expressed by the matrix formula

$$S' A \bar{S} = A, \tag{10}$$

in which A is the matrix of the coefficients a_{ik} of H, $\bar{S}$ is the matrix obtained from S by changing every element into its conjugate value, and S' is obtained from S by interchanging rows and columns. In forming the product the elements of the i-th row of S' are multiplied into the elements of the k-th column of A to form the element of the i-th row and k-th column of the product matrix. The resulting matrix is combined in a similar manner with $\bar{S}$. That (10) is the condition for the invariance of H may readily be seen by substituting $z_i' = \sum\limits_{\lambda} c_{i\lambda} z_\lambda$, $\bar{z}_k' = \sum \bar{c}_{k\mu} \bar{z}_\mu$ in $H' = \sum a_{ik} z_i' \bar{z}_k'$ and making the matrix of the result identical with A.*

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* The substitution S can be expressed in but one way in the form $(\bar{A} + \bar{T})^{-1} (\bar{A} - \bar{T})$ in which T is subject to the restriction $T + \bar{T}' = 0$. The sum $A + T$ means the matrix whose elements are the sums of the corresponding elements in A and T. It may readily be shown that any substitution of this form changes H into itself. For, the matrix equation (or identity)

$$\bar{A}' + \bar{T}' + T + TA^{-1}\bar{T}' = \bar{A}' - \bar{T}' - T + TA^{-1}\bar{T}'$$

may (by use of $\bar{A}' = A$, the condition that H is an Hermitian form) be written

$$(1 + TA^{-1})(\bar{A}' + \bar{T}') = (1 - TA^{-1})(\bar{A}' - \bar{T}')$$

or

$$\bar{U}A^{-1}U' = \bar{V}A^{-1}V',$$

Using the form (9) for S, the product (10) is

$$\begin{vmatrix} S_{11}, & S_{12}, & \dots, & S_{1,\mu+2} \\ S_{21}, & S_{22}, & \dots, & S_{2,\mu+2} \\ \dots & \dots & \dots & \dots \\ S_{\mu+2,1}, & S_{\mu+2,2}, & \dots, & S_{\mu+2,\mu+2} \end{vmatrix}, \qquad (11)$$

in which S_{ik} represents a matrix of e_i rows and e_k columns constructed in the following manner. Let $\mathbf{M}'_i$ denote the matrix obtained from S' by replacing all its elements by zeros except those in M'_i, and let $\overline{\mathbf{M}}_k$ be formed similarly from $\bar{S}$. Then construct the product $\mathbf{M}'_i A \overline{\mathbf{M}}_k$ in the same way as described for (10). The non-zero elements of this product form a matrix of e_i rows and e_k columns which is denoted by S_{ik}. If now we impose the condition that the matrix (11) is identical with A, the matrices S_{ik} take the special forms

$$\left[\begin{array}{ccccc} 0 & 0 & \dots\dots & & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots\dots & \dots\dots & 0 & a_{\alpha+r_i-r_k,\,\beta+r_k} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & a_{\alpha+r_i-2,\,\beta+2} & & \dots\dots \\ 0 & a_{\alpha+r_i-1,\,\beta+1} & \dots\dots & & \dots\dots \\ a_{\alpha+r_i,\,\beta} & a_{\alpha+r_i,\,\beta+1} & \dots\dots & & a_{\alpha+r_i,\,\beta+r_k} \end{array}\right|$$

in which $\alpha=e_1+\dots+e_{i-1}$, $\beta=e_1+\dots+e_{k-1}$, $r_i=e_i-1$, $r_k=e_k-1$, and i, k do not exceed μ, $i<k$;

$$S_{ki}=\begin{vmatrix} 0 & \dots\dots\dots\dots & & 0 & a_{\alpha,\,\beta+r_k} \\ 0 & \dots\dots\dots & 0 & a_{\alpha+1,\,\beta+r_k-1} & a_{\alpha+1,\,\beta+r} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots\dots\ 0 & a_{\alpha+r_i,\,\beta+r_k-r_i} & \dots\dots & a_{\alpha+r_i,\,\beta+r_k} \end{vmatrix},$$

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in which $\bar{U}=A+T$, $V=A-T$. By taking the inverse of both members, we have

$$(U')^{-1}A\bar{U}^{-1}=(V')^{-1}A\bar{V}^{-1},$$

whence $V'(U')^{-1}A\bar{U}^{-1}\bar{V}=A$, or $(U^{-1}V)'A(\bar{U}^{-1}\bar{V})=A$. By comparison with (10) we see that $U^{-1}V$ or $(\bar{A}+\bar{T})^{-1}(\bar{A}-\bar{T})$ transforms H into itself.

On the other hand, if S is the matrix of any substitution which transforms H into itself, then a matrix T is uniquely defined by the formula $T=A(E-\bar{S})(E+\bar{S})^{-1}$, in which E is the unit matrix. The solution of this equation for S gives $S=(\bar{A}+\bar{T})^{-1}(\bar{A}-\bar{T})$.

Moreover, $\bar{T}'=(E+S')^{-1}(E-S')A=(E+S')^{-1}(E+S'-2S')A=A-2(E+S')_{-1}S'A$. But we have $T=A(2E-E-\bar{S})(E+\bar{S})^{-1}=2A(E+\bar{S})^{-1}-A=2A(E+A^{-1}S'^{-1}A)^{-1}-A$ (using (10)) $=2[(E+A^{-1}S'^{-1}A)A^{-1}]^{-1}-A=2(A^{-1}+A^{-1}S'-)^{-1}-A=2[A^{-1}S'^{-1}(S'+E)]^{-1}-A=2(E+S')^{-1}S'A-A$. Hence $T+\bar{T}'=0$.

This proof is given, in a somewhat different form, by A. Loewy in *Nova Acta der Kaiserl. Leop.-Carol. Deutschen Akademie der Naturforscher*, Vol. LXXI (1898), § 2.

$$S_{\iota,\mu+1} = \left| \begin{array}{ccc} 0 & \ldots\ldots\ldots\ldots & 0 \\ \ldots & \ldots\ldots\ldots\ldots & \ldots \\ 0 & \ldots\ldots\ldots\ldots & 0 \\ a_{\alpha+r_\iota,\beta} & \ldots\ldots & a_{\alpha+r_\iota,\beta+\mu-1} \end{array} \right|, \quad S_{\mu+1,\iota} = \left| \begin{array}{cccc} 0 & \ldots\ldots\ldots & 0 & a_{\alpha,\beta+r_\iota} \\ 0 & \ldots\ldots\ldots & 0 & a_{\alpha+1,\beta+r_\iota} \\ \ldots & \ldots\ldots\ldots & \ldots & \ldots \\ 0 & \ldots\ldots\ldots & 0 & a_{\alpha+\mu-1,\beta+r_\iota} \end{array} \right|$$

and $S_{\iota,\mu+2}$, $S_{\mu+2,\iota}$ have all elements zero. We observe in particular that when $e_\iota = e_k$, $S_{\iota k}$ is a square matrix having zero elements above the second diagonal and hence but one non-vanishing element in the first row. If $e_\iota > e_k$, all the elements of the first row are zero. The non-zero elements satisfy certain conditions which it is not necessary to specify.

If now we assume $e_1 > e_2$, it follows from the remark just made concerning the first row of $S_{\iota k}$ and from the conditions $e_2 \gtreqless e_3 \ldots.$ that all the elements of the first row of A must be zero except a_{1,e_1}. Hence the minor A_{e_1,e_1} is zero and by applying transformation (II) we express H in the form

$$\eta_1\bar{\eta}_1 - \eta_{e_1}\bar{\eta}_{e_1} + H_2,$$

in which H_2 contains only $n-2$ variables, whose coefficients are the same as the minor of A obtained by erasing the first and the e_1-th rows and columns. If A_2, the matrix of the coefficients in H_2, contains but one non-vanishing element in its first row, it will be a_{2,e_1-1}. This happens when $e_1 - 1 > e_2$. We repeat the transformation (II) on H_2, obtaining

$$H_2 = \eta_2\bar{\eta}_2 - \eta_{e_1-1}\bar{\eta}_{e_1-1} + H_4.$$

This process is to be continued as long as the first row of each successive matrix A, A_2, A_4, $\ldots.$ contains but one non-vanishing element. If $e_1 = e_2 + \varepsilon$ $(\varepsilon \gtreqless 0)$, the matrix $A_{2\varepsilon}$ contains the least two non-zero elements in its first row, namely $a_{\varepsilon+1,e_2}$ and $a_{\varepsilon+1,e_1+e_2}$. We reduce this second coefficient to zero by substituting for the variables* ζ_{e_2}, $\zeta_{e_1+e_2}$ in $H_{2\varepsilon}$ the expressions

$$\zeta_{e_2} = \eta_{e_2} + \bar{a}_{\varepsilon+1,e_1+e_2}\,\eta_{e_1+e_2}, \quad \zeta_{e_1+e_2} = -\bar{a}_{\varepsilon+1,e_2}\,\eta_{e_1+e_2}. \tag{12}$$

If other non-vanishing elements occur in the same row, they can be reduced to zero in like manner. It is to be observed that if any coefficient $a_{\iota k}$ is changed by this transformation into $b_{\iota k}$, its conjugate $a_{k\iota}$ is at the same time changed into $b_{k\iota} = \bar{b}_{\iota k}$.

We now continue with the transformation (II), it being necessary after each application to reduce all the elements in the first row of each remaining matrix to zero except the one element which occurs in S_{11}. If e_1 is even, this process can be continued until all the rows and columns of A which pass through S_{11} have disappeared. If e_1 is odd, there will be one row and column left, containing the common element $a_{\alpha\alpha}\,(\alpha = \frac{1}{2}(e_1 + e_2))$. After reducing all the elements of this row and column to zero except $a_{\alpha\alpha}$ by means of (12), we make the transformation (I) in which m is replaced by α, and $\varkappa$ runs from $\alpha + 1$ to n. One more term is thus obtained for the canonical form, the sign of which is unknown. In this manner we have replaced the first e_1 variables $\zeta_1, \ldots, \zeta_{e_1}$ of the form H by new variables which are reduced to the canonical form. The number of terms in this group which are certainly positive is $e_1/2$ or $(e_1-1)/2$ according as e_1 is even or odd. These numbers can be expressed by the single formula $\frac{1}{4}[2e_1 - 1 - (-1)^{e_1}]$.

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* To avoid multiplicity of notation we shall repeatedly use the letters ζ and η to denote the variables before and after any transformation.

We proceed to reduce the next set of e_2 variables $\zeta_{e_1+1}, \ldots, \zeta_{e_1+e_2}$ in the same way, and so on. The final result may be expressed in the following theorem:

If the characteristic equation $C = 0$ of the substitution S has an m-fold root $\rho\,(\rho\bar{\rho} = 1)$, and if the exponents of the corresponding elementary divisors are e_1, e_2, $\ldots$, $(e_1 + e_2 + \ldots = m)$, then these numbers are restricted by the condition

$$\frac{1}{4}\sum_i [2e_i - 1 + (-1)^{e_i}] \leqq p, \tag{13}$$

since the left member gives the number of terms which are certainly positive after H has been reduced to the canonical form.

This summation may extend to all the exponents e, since in case any of them are equal to unity the corresponding terms in (13) vanish.

We turn now to the case in which the m equal roots ρ are not unimodular. Let σ denote the root of $C = 0$ which satisfies the relation $\rho\bar{\sigma} = 1$, and let the exponents of the elementary divisors corresponding to this root be e'_1, e'_2, $\ldots$, $(\Sigma e'_i = m'$, m' being the number of equal roots $\sigma)$. As will be seen presently, the numbers e'_i and e_i coincide. Suppose S reduced to the normal form

$$\left|\begin{matrix} M_1 & & & & \\ & \ddots & & & \\ & & M_{\mu+1} & & \\ & & & N_1 & \\ & & & & \ddots \\ & & & & & N_{\nu+2} \end{matrix}\right|$$

in which M_i has the form indicated in (9), N_i is a similar matrix in σ of order e'_i, and $N_{\nu+2}$ is expressible in terms of the remaining roots of the characteristic equation.

The product $S' A \bar{S}$ will be made up of matrices obtained from $\mathbf{M}_i' A \overline{\mathbf{M}}_k$, $\mathbf{M}_i' A \overline{\mathbf{N}}_k$, $\mathbf{N}_i' A \overline{\mathbf{N}}_k$, etc., as described in connection with (11) except that in $\mathbf{M}_i' A \overline{\mathbf{M}}_k$, $\mathbf{N}_i' A \overline{\mathbf{N}}_k$* every element is multiplied by $\rho\bar{\rho}$, $\sigma\bar{\sigma}$ respectively, and since these multipliers are not unity the elements all vanish when condition (10) is imposed. The same is true of all the matrices whose symbols contain the letter $\mathbf{N}_{\nu+2}$ excepting only $\mathbf{N}_{\nu+2}' A \overline{\mathbf{N}}_{\nu+2}$. Assume as before that the exponents e_i (and e_i') are arranged in descending order of magnitude.

Suppose now $e_1 > e_1'$. Then the matrices $\mathbf{M}_1' A \overline{\mathbf{N}}_k$ have all the elements of the first row zero and the determinant $S' A \bar{S}$ would vanish. Moreover, we can not have $e_1' > e_1$, since then the first column of $S' A \bar{S}$ would consist only of zeros. Hence $e_1' = e_1$. If $e_1 > e_i$ or e_i' ($i \geqq 2$), the only non-zero element in the first row of A is $a_{1\beta}$, $\beta = m + e_1$. Otherwise there are additional non-zero elements in this row which we reduce to zero by means of the transformation (12). Supposing this has been done, we observe that the element $a_{\beta\beta}$ occurs in the matrix $\mathbf{N}_1' A \overline{\mathbf{N}}_1$, and is therefore zero. Hence in transformation (II), which we now apply, the term a_{mm} is zero. The form H then becomes

$$\eta_1\bar{\eta}_1 - \eta_\beta\bar{\eta}_\beta + H_2, \quad H_2 = \sum_{\kappa,\lambda} a_{\kappa\lambda}\,\eta_\kappa\bar{\eta}_\lambda \quad (\kappa, \lambda \neq 1, \beta).$$

The determinant A_2 of H_2 is the minor of A obtained by striking out the first and the β-th rows and columns.

The above process is repeated on H_2, giving

$$H_2 = \eta_2\bar{\eta}_2 - \eta_{\beta-1}\bar{\eta}_{\beta-1} + H_4.$$

This may be continued until the variables with subscripts $1, 2, \ldots, e_1, \beta, \beta-1, \ldots, \beta - e_1 + 1$ have been replaced by new variables in the canonical form

$$\sum_{i=1}^{e_1}(\eta_i\bar{\eta}_i - \eta_{\beta+1-i}\bar{\eta}_{\beta+1-i}).$$

The remaining portion of H, which we will denote by H_{2e_1}, has for the matrix of its coefficients the minor A_{2e_1} of A obtained by erasing all the rows and columns meeting in the matrices $\mathbf{M}_1' A \overline{\mathbf{N}}_1$, $\mathbf{N}_1' A \overline{\mathbf{M}}_1$.

We may now prove by a repetition of the argument used above that $e_2' = e_2$, since otherwise A_{2e_1} would vanish. We next reduce H_{2e_1} to a sum of e_2 positive and e_2 negative terms plus a form $H_{2(e_1+e_2)}$ the matrix of whose coefficients is the minor obtained from A_{2e_1} by erasing all the rows and columns meeting in the matrices $\mathbf{M}_2' A \overline{\mathbf{N}}_2$, $\mathbf{N}_2' A \overline{\mathbf{M}}_2$. This process is to be continued as long as there are e's left which are greater than unity. The coefficients of the remaining terms of H which are not affected by the preceding transformations form the matrix

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* It will be convenient to use these symbols to denote the matrices of e_i rows and e_k' columns formed by omitting the zeros.

$$\left| \begin{array}{ccc} 0 & \mathbf{M}_{\mu+1}' A \overline{\mathbf{N}}_{\mu+1} & 0 \\ \mathbf{N}_{\mu+1}' A \overline{\mathbf{M}}_{\mu+1} & 0 & 0 \\ 0 & 0 & \mathbf{N}_{\mu+2}' A \overline{\mathbf{N}}_{\mu+2} \end{array} \right|,$$

since by a repetition of a preceding argument we evidently have $\nu = \mu$. Now reduce all but one of the terms in the first row of this matrix to zero by means of (12) and then apply (II), and so proceed until all the elements of $\mathbf{M}_{\mu+1}' A \overline{\mathbf{N}}_{\mu+1}$ have disappeared.

We thus obtain an expression for H in which the first $2m$ variables have been reduced to the canonical form, the terms containing them being alternately positive and negative. There are m positive terms and hence in this case $m \leqq p$.

Suppose now that the characteristic equation $C = 0$ has m_1 equal roots ρ_1, m_2 equal roots $\rho_2, \ldots, m_k$ equal roots ρ_k, none of these being unimodular and $m_1, m_2, \ldots, m_k \geqq 0$; also the same number of equal roots $\sigma_1, \sigma_2, \ldots, \sigma_k$ such that $\rho_i \bar{\sigma}_i = 1$. Suppose further that there are μ_1 equal unimodular roots τ_1, μ_2 equal roots $\tau_2, \ldots, \mu_\kappa$ equal roots τ_κ $(\mu_1, \ldots, \mu_\kappa \geqq 1)$, and let $e_{i1}, e_{i2}, \ldots$ denote the exponents of the elementary divisors corresponding to the root τ_i. If the substitution S be expressed by means of suitably chosen variables in the normal form corresponding to (9), it is evident that when H is also expressed in terms of the same variables it consists of groups of terms, the first group containing only the variables $\zeta_1, \ldots, \zeta_{2m_1}$, the next group containing only $\zeta_{2m_1+1}, \ldots, \zeta_{2(m_1+m_2)}$, and so on to the $(k+1)$-th group containing only the variables $\zeta_\alpha, \ldots, \zeta_{\alpha+\mu_1}$ $(\alpha = 2m_1 + \ldots + 2m_k)$, etc. It is further clear that each group of terms can be reduced separately to the canonical form.* Counting up the total number of terms which are certainly positive we obtain, as a restriction on the possible number of equal roots, the formula

$$\sum_i m_i + \tfrac{1}{4} \sum_{\alpha, \beta} \left[2 e_{\alpha\beta} - 1 + (-1)^{e_{\alpha\beta}} \right] \leqq p, \quad \Sigma\, 2 m_i + \Sigma\, e_{\alpha\beta} = n.$$

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* The reduction of H to the canonical form is virtually contained in the processes and results of the *Acta Nova* memoir of Loewy previously quoted. It is believed that the different treatment presented above will be found advantageous in various respects.

§ 5. *The Fixed Points when C Has Equal Roots.*

To determine the fixed points of a substitution whose characteristic equation has m equal roots ρ, consider S in the normal form (9). The fixed points corresponding to the root ρ are determined by the equations $\zeta_i' = \rho\,\zeta_i$. These reduce to

$$\zeta_2 = \zeta_3 = \ldots\ldots = \zeta_{e_1} = 0;\quad \zeta_{e_1+2} = \zeta_{e_1+3} = \ldots\ldots = \zeta_{e_1+e_2} = 0;\quad \ldots\ldots;$$
$$\zeta_k = 0 \quad (k = m+1,\ m+2,\ \ldots\ldots,\ n).$$

Each of these equations is equivalent to two equations in real variables. They therefore determine a flat space of

$$2(n-1) - 2\,[\Sigma\,(e_i - 1) + n - m] = 2\,[m - 1 - \Sigma\,(e_i - 1)]$$

dimensions. But since $\Sigma\, e_i = m$, this number reduces to $2(\lambda - 1)$, in which λ is the number of elementary divisors containing the multiple root ρ. From this result we deduce the evident generalization:

If $C = 0$ *has* m_1 *equal roots* ρ_1, m_2 *equal roots* ρ_2, $\ldots\ldots$, m_s *equal roots* ρ_s, *then the fixed points of* S *constitue* s *flat spaces of dimensions* $2(\lambda_1 - 1)$, $2(\lambda_2 - 1)$, $\ldots\ldots$, $2(\lambda_s - 1)$ *respectively, in which* λ_i *is the number of elementary divisors containing the root* ρ_i. *None of these* s *loci have points in common.*

This theorem includes the simple roots, since the corresponding values of λ are 1 and $2(\lambda - 1)$ has the required value zero for such cases.

CORNELL UNIVERSITY, *October*, 1909.

The Osculants of Plane Rational Quartic Curves.

By H. I. Thomsen.

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§ 1. *Cases of Degeneracy.*

We take the equations of the rational plane curve in the form

$$x_i = (\alpha_i t)^n \equiv a_i t^n + b_i t^{n-1} + \ldots + p_i, \qquad i = 1, 2, 3,$$

the arrowhead in $(\alpha_i t)^n$ indicating the absence of binomial coefficients.

These equations in general represent a curve of order n. The order of the curve will, however, be less than n if:

1) There is a factor involving t common to the three $(\alpha_i t)^n$;

2) It is possible by a substitution of the kind*

$$t' = \frac{(\beta t)^m}{(\gamma t)^m}$$

to reduce the equations to the form

$$x_i = (\alpha_i t)^{\frac{n}{m}}.$$

In the latter case the original equations represent the curve of order $\frac{n}{m}$ repeated m times.

The case of a linear relation existing between the three x_i is included in this, since the equations

$$x_1 = (\alpha_1 t)^n, \quad x_2 = (\alpha_2 t)^n, \quad x_3 = k_1 (\alpha_1 t)^n + k_2 (\alpha_2 t)^n$$

are equivalent to

$$x_1 = t', \quad x_2 = 1, \quad x_3 = k_1 t + k_2,$$

where

$$t' = \frac{(\alpha_1 t)^n}{(\alpha_2 t)^n}.$$

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* Where $\frac{n}{m}$ is a positive integer.

We postulate that, after all common factors are removed, the equations can not represent the product of distinct curves. It may be observed that a curve of genus zero is not necessarily rational. Thus the product of a conic and a cubic of genus one is a quintic of genus zero, but is not rational.[*]

For brevity we sometimes refer to the point of the curve given by $t = t_1$ as the point t_1.

We may make the equations homogeneous in the parameter by writing $\frac{t}{t'}$ for t. Then we may define the m-th osculant of the curve at the point $\frac{\tau}{\tau'}$ as the curve represented by the equations

$$x_i = (\tau D_t + \tau' D_{t'})^m (\alpha_i t)^n,$$

in which we may let $t' = \tau' = 1$, if desired. It is well known that osculants at the point τ touch the curve there, that all first osculants touch the stationary tangents of the curve, and that the $(n-1)$-st osculant is the tangent to the curve.

We shall see that osculants degenerate under certain conditions. As an example we may take the quartic

$$x_1 = t^4 + b t^3, \quad x_2 = t + e, \quad x_3 = t^2.$$

This curve is bicuspidal, having cusps at the points (0, 1, 0) and (1, 0, 0). If $b = 4e$, it has a third cusp at the point $(16 e^4, e, -4 e^3)$ given by $t = -2e$, the tangent at this point, $x_1 - 16 e^3 x_2 = 0$, meeting the curve again at the point $(48 e^4, 3e, 4e^2)$ given by $t = 2e$.

The first osculant of the curve at the point τ is

$$x_1 = (4\tau + b) t^3 + 3 b \tau t^2, \quad x_2 = 3t + \tau + 4e, \quad x_3 = 2t^2 + 2\tau t.$$

Let us first let $e = 1$, and form the osculant at the point $\tau = 2$. It is

$$x_1 = (8 + b) t^3 + 6 b t^2, \quad x_2 = 3(t + 2), \quad x_3 = 2t(t + 2).$$

The elimination[†] of t gives

$$4 x_1 x_2 (3 x_3 + 4 x_2) - 9 x_3^2 ((8 + b) x_3 + 4 b x_2) = 0,$$

which becomes, if $b = 4$,

$$4 (x_1 x_2 - 9 x_3^2)(4 x_2 + 3 x_3) = 0.$$

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[*] Clebsch-Lindemann: "Vorlesungen über Geometrie," Vol. I, p. 884.

[†] By "the elimination of t" we mean its elimination from two equations such as

$$\frac{x_1}{x_2} = \frac{(8 + b) t^3 + 6 b t^2}{3(t + 2)} \text{ and } \frac{x_1}{x_3} = \frac{(8 + b) t^3 + 6 b t^2}{2t(t + 2)}.$$

If we let, however, $b=4$ in the quartic and form the osculant at $\tau=2$, we have

$$x_1 = 12\,t^2(t+2), \quad x_2 = 3\,t + (2+4\,e), \quad x_3 = 2\,t\,(t+2),$$

or, on eliminating t,

$$x_1 x_2 (x_1 + 12\,x_3) - 9\,x_3^2 (x_1 + 2\,(2+4\,e)\,x_3) = 0,$$

which becomes, if $e=1$,

$$(x_1 x_2 - 9\,x_3^2)(x_1 + 12\,x_3) = 0.$$

The lines $4\,x_2 + 3\,x_3 = 0$ and $x_1 + 12\,x_3 = 0$ meet the conic $x_1 x_2 - 9\,x_3^2 = 0$ at the point $(48, 3, -4)$ which is on the cusp tangent $x_1 - 16\,x_2 = 0$.

It seems, then, that this conic and any line on this point form an osculant cubic of the quartic at the point $t=2$. The first osculant at the point τ of

$$x_1 = t^4 + 4\,t^3, \quad x_2 = t+1, \quad x_3 = t^2$$

being

$$x_1 = 4\,(\tau+1)\,t^3 + 12\,\tau\,t^2, \quad x_2 = 3\,t + \tau + 4, \quad x_3 = 2\,t^2 + 2\,\tau\,t,$$

the first osculant of the latter curve at the point τ_1 is

$$\begin{aligned} x_1 &= 12\,(\tau\tau_1 + \tau + \tau_1)\,t^2 + 24\,\tau\tau_1\,t, \\ x_2 &= 6\,t + 3\,(\tau + \tau_1) + 12, \\ x_3 &= 2t^2 + 4\,(\tau + \tau_1)\,t + 2\,\tau\tau_1. \end{aligned}$$

If in the first of these curves we let $\tau = 2$, we have $\tau + 2 = 0$ and

$$x_1 = 12\,t^2, \quad x_2 = 3, \quad x_3 = 2\,t.$$

If in the second curve we let $\tau = \tau_1 = 2$, we have

$$x_1 = 96\,t\,(t+1), \quad x_2 = 6\,(t+4), \quad x_3 = 2\,(t^2 + 8\,t + 4).$$

Hence the conic which forms part of the degenerate first osculant is not the second osculant.

It may be noted that osculants of degenerate curves are not, in general, degenerate. Thus the first osculant of

$$x_1 = t^3, \quad x_2 = 3\,t^2, \quad x_3 = 3\,t$$

is

$$x_1 = \tau\,t^2, \quad x_2 = t^2 + 2\,\tau\,t, \quad x_3 = 2\,t + \tau,$$

and that of the repeated conic

$$x_1 = t^4, \quad x_2 = 2\,t^2, \quad x_3 = 1$$

is

$$x_1 = \tau t^3, \quad x_2 = t^2 + \tau t, \quad x_3 = 1.$$

The osculant cubics of

$$x_1 = t^4 + 4t^3, \quad x_2 = t + e, \quad x_3 = t^2$$

being

$$x_1 = 4(\tau + 1)t^3 + 12\tau t^2, \quad x_2 = 3t + \tau + 4e, \quad x_3 = 2t^2 + 2\tau t,$$

the locus of the nodes of these cubics is, by the formula of page 226, the sextic

$$x_1 = \tau^3(\tau + 4e)(\tau - 2)^2, \quad x_2 = (\tau + 1)(\tau - 2e)^3, \quad x_3 = \tau^2(\tau - 2)(\tau - 2e).$$

If $e = 1$, this becomes $(\tau - 2)^2 = 0$ and

$$x_1 = \tau^3(\tau + 4), \quad x_2 = \tau + 1, \quad x_3 = \tau^2,$$

which is exactly the form which the equation of the original quartic takes for this value of e. Hence we see that for a tricuspidal quartic part of the locus of nodes of osculant cubics is the quartic itself. It remains to find a meaning for the factor $(t - 2)^3$. If we write

$$\rho \equiv \frac{x_1}{x_3} = \frac{\tau(\tau + 4e)(\tau - 2)}{\tau - 2e}, \quad \sigma \equiv \frac{x_2}{x_3} = \frac{(\tau + 1)(\tau - 2e)}{\tau^2(\tau - 2)},$$

we have

$$\tau^3 + (4e - 2)\tau^2 - (8e + \rho)\tau + 2e\rho = 0$$

and

$$\sigma\tau^3 - (2\sigma + 1)\tau^2 + (2e - 1)\tau + 2e = 0,$$

or, if $\tau = \tau_1 + 2$,

$$\tau_1^3 + 4(e + 1)\tau_1^2 + (8e + 4 - \rho)\tau_1 + 2\rho(e - 1) = 0,$$
$$\sigma\tau_1^3 + (4\sigma - 1)\tau_1^2 + (4\sigma + 2e - 5)\tau_1 + 6(e - 1) = 0,$$

the resultant of which equations gives the trilinear equation of the sextic.

The resultant of the equations*

$$at^3 + bt^2 + ct + d = 0 \quad \text{and} \quad a_1t^3 + b_1t^2 + c_1t + d_1 = 0$$

is

$$|ad_1|^3 - 2|ad_1|\,|ab_1|\,|cd_1| - |ad_1|\,|ac_1|\,|bd_1| + |ac_1|^2|cd_1| + |bd_1|^2|ab_1| - |ab_1|\,|bc_1|\,|cd_1| = 0.$$

In our case $e - 1$ is a factor of both d and d_1; therefore it is a factor of each of the determinants in which d occurs, and is a factor of the resultant. If we remove this factor, the remaining factor does not vanish identically if $e = 1$, but becomes

$$\frac{|cd_1|}{e - 1}\Big[|ac_1|^2 - |ab_1|\,|bc_1|\Big] = 0.$$

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* Salmon-Fiedler: "Alg. der Lin. Trans.," p. 93.

If $e = 1$,

$$\frac{|c\,d_1|}{e-1} \equiv \begin{vmatrix} 12-\rho, & 2\rho \\ 4\sigma-3, & 6 \end{vmatrix} = -8\,(\sigma\rho - 9),$$

or this factor gives $x_1 x_2 - 9\,x_3^2 = 0$.

In like manner

$$|a\,c_1|^2 - |a\,b_1|\;|b\,c_1| = 0$$

gives

$$(x_1 x_2 - x_3^2)^2 - x_3^2\,(x_1 + 4\,x_3)\,(4\,x_2 + x_3) = 0,$$

which, as may easily be verified, is the original quartic.

Hence it appears that since the conic $x_1 x_2 - 9\,x_3^2 = 0$ *and any line on the point* $(48,\ 3,\ -4)$ *form an osculant cubic at* $t = 2$, *any point of this conic is the node of an osculant cubic and the conic is a part of the nodal locus, corresponding to the factor* $(t-2)^2$ *in the parametric equation of that locus.*

We meet with degeneracies of the kind now being considered when we form the line equations of cuspidal curves. Thus the cubic

$$x_1 = t^3, \quad x_2 = 1, \quad x_3 = t^2 + c\,t$$

becomes cuspidal if $c = 0$. The equation of its tangent is

$$(2\,t + c)\,x_1 + t^3\,(t + 2\,c)\,x_2 - 3\,t^2\,x_3 = 0,$$

and its line equations are

$$\xi_1 = 2\,t + c, \quad \xi_2 = t^3(t + 2\,c), \quad \xi_3 = -3\,t^2.$$

If $c = 0$, this becomes $t = 0$ and

$$\xi_1 = 2, \quad \xi_2 = t^3, \quad \xi_3 = -3\,t,$$

or

$$27\,\xi_1^2\,\xi_2 + 4\,\xi_3^3 = 0.$$

The elimination of t from the equations of the quartic gives

$$(9\,\xi_1\,\xi_2 - c\,\xi_3^2)^2 - 4\,\xi_3^2\,(3\,c\,\xi_1 - \xi_3)\,(3\,\xi_2 - c^2\,\xi_3) = 0.$$

The points of inflexion are

$$\xi_1 = 0, \quad \xi_2 = 0 \text{ and } c^3\,\xi_1 - \xi_2 = 0,$$

and the node is

$$c^3\,\xi_1 + \xi_2 - c^2\,\xi_3 = 0.$$

If $c = 0$, the quartic becomes

$$3\,\xi_2\,(27\,\xi_1^2\,\xi_2 + 4\,\xi_3^3) = 0.$$

Two points of inflexion and the node are $\xi_2 = 0$, which point is a part of the quartic, and is also the cusp of the cuspidal cubic which forms the other part.

The pencil of conics on the three stationary tangents and the tangent at $t = 1$ of the quartic is

$$(3\xi_2 - c^3\xi_3)(3\xi_1 + (2 + c)\xi_3) + \tau[(3c\xi_1 - \xi_3)(3\xi_2 + (1 + 2c)\xi_3)] = 0,$$

and the eighth common line is given by

$$2t + c(t + 1) - \tau(t + 1 + 2c) = 0.$$

If $c = 0$, this becomes

$$2t - \tau(t + 1) = 0.$$

If $t = 0$, $\tau = 0$ and for this value of τ, the conic of the pencil is

$$3\xi_2(3\xi_1 + 2\xi_3) = 0;$$

i. e., the point $\xi_2 = 0$ and the meet of the stationary tangent of the cubic and the tangent at $t = 1$.

In this case we find that, when we give the parameter an evident geometrical meaning, one part of the degenerate curve corresponds to a particular value of the parameter.

We now give an example of a conic degenerating into a repeated line. The osculant conic at the point τ of the cubic

$$x_1 = 3t^2, \quad x_2 = 3t, \quad x_3 = t^3 - 1$$

is

$$x_1 = t^2 + 2\tau t, \quad x_2 = 2t + \tau, \quad x_3 = \tau t^2 - 1.$$

If $\tau = 1$, this becomes

$$x_1 = t^2 + 2t, \quad x_2 = 2t + 1, \quad x_3 = t^2 - 1 \equiv t^2 + 2t - (2t + 1),$$

or, if $t' = \dfrac{t^2 + 2t}{2t + 1}$,

$$x_1 = t', \quad x_2 = 1, \quad x_3 = t' - 1.$$

Hence the osculant at this point, which is a point of inflexion of the cubic, is the stationary tangent $x_1 - x_2 - x_3 = 0$, repeated.

The tangent to the cubic at the point τ_1 meets the stationary tangent, $x_1 - x_2 - x_3 = 0$, at a point P. The other tangents to the curve from P are the stationary tangent counted twice and an additional line touching the curve at a point τ_2. It may easily be shown that τ_1 and τ_2 form a pair of the involution defined by $2\tau_1\tau_2 + \tau_1 + \tau_2 + 2 = 0$, and that τ_1 and τ_2 give the same point on the repeated line. The double points of the involution are given by $\tau^2 + \tau + 1 = 0$, and ω and ω^2 give* on the line the points where the two other stationary tangents meet it. We shall see later† that the line equation of this osculant takes the form of the product of the equations of these two points.

The result of eliminating t from the equations of an osculant is

$$(\tau^2 x_1 - x_2 - \tau x_3)^2 - 4(\tau x_1 - \tau^2 x_2 - x_3)(x_1 - \tau x_2 - \tau^2 x_3) = 0,$$

which becomes, if $\tau = 1$,

$$-3(x_1 - x_2 - x_3)^2 = 0.$$

The four points of intersection, other than the point of contact, of the cubic and the osculant conic are given by

$$\tau^2 t^4 - 4t^3 + 6\tau t^2 - 4\tau^2 t + 1 = 0,$$

for which quartic the invariant $g_2 = 0$, its roots forming a self-apolar set.

Lüroth‡ has shown how to reduce the equation of a repeated curve. Brill§ states certain conditions under which a quartic degenerates, and gives an interesting note in regard to degeneracies due to the occurrence of a common factor.

§ 2. *The Fundamental Involution.*

The equations of the curve being $x_i = (\alpha_i t)^n$, we can find $n - 2$ linearly independent forms of the n-th order apolar to the three $(\alpha_i t)^n$.

If $(\beta_0 t)^n, \ldots, (\beta_{n-3} t)^n$ are $n - 2$ such forms, the equation

$$\lambda_0(\beta_0 t)^n + \ldots + \lambda_{n-3}(\beta_{n-3} t)^n = 0$$

defines the fundamental involution of the curve.‖ If we assume $n - 3$ roots of this equation, the remaining three are fixed. There is no fundamental involution for the conic unless it is degenerate.

For the cubic it is well known that the roots of $(\beta_0 t)^3 = 0$ are the parameters of the points of inflexion.

The condition that

$$t^4 - s_1 t^3 + s_2 t^2 - s_3 t + s_4 = 0$$

be apolar to

$$a_i t^4 + b_i t^3 + c_i t^2 + d_i t + e_i = 0$$

is

$$a_i s_4 + b_i \frac{s_3}{4} + c_i \frac{s_2}{6} + d_i \frac{s_1}{4} + e_i = 0,$$

which is the result of completely polarizing the latter equation.

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* ω and ω^2 being cube roots of unity.

† Page 228.

‡ *Math. Annalen*, Vol. IX, p. 163.

§ *Math. Annalen*, Vol. XII, p. 101.

‖ Stahl: *Crelle*, Vol. CI, p. 300. Meyer: "Apolarität und Rationale Curven."

Hence we can write the fundamental involution of the quartic in the form:

$$\begin{vmatrix} t^4+s_4 & a_1 s_4+e_1 & a_2 s_4+e_2 & a_3 s_4+e_3 \\ -t^3 & \frac{d_1}{4} & \frac{d_2}{4} & \frac{d_3}{4} \\ t^2 & \frac{c_1}{6} & \frac{c_2}{6} & \frac{c_3}{6} \\ -t & \frac{b_1}{4} & \frac{b_2}{4} & \frac{b_3}{4} \end{vmatrix} = 0.$$

This form of the involution might have to be modified for special cases, but it is obvious that the fundamental involution of any rational curve may be found by the method here used.

We shall call the fundamental involution the I_n.

If a curve is given by $x_i = (\alpha_i t)^n$, using for the moment binomial coefficients, the three conditions that the n parameters $(t_1, t_2, \ldots, t_n)$ be a set of the I_n are

$$(\alpha_i t_1)(\alpha_i t_2) \ldots (\alpha_i t_n) = 0.$$

The first osculant at the point t_1 is $x_i = (\alpha_i t_1)(\alpha_i t)^{n-1}$, and the conditions that $(t_2, t_3, \ldots, t_n)$ be a set of the I_{n-1} of this osculant are again

$$(\alpha_i t_1)(\alpha_i t_2) \ldots (\alpha_i t_n) = 0.$$

Hence any set of the I_{n-1} on the osculant at t_1 form, with t_1, a set of the I_n on the original curve.

Thus for the quartic, as Stahl has shown, t_1 and the parameters of the points of inflexion of the osculant cubic at t_1 form a set of the I_4. In general, if we select the $n-3$ parameters $(t_1, t_2, \ldots, t_{n-3})$, the other three members of the set of the I_n which these determine are the parameters of the points of inflexion of the common osculant cubic

$$x_i = (\alpha_i t)^3 (\alpha_i t_1)(\alpha_i t_2) \ldots (\alpha_i t_{n-3}).$$

By a common osculant we mean a curve such as $x_i = (\alpha_i t_1)(\alpha_i t_2)(\alpha_i t)$ which is evidently both the tangent at t_2 of $x_i = (\alpha_i t_1)(\alpha_i t)^2$ and the tangent at t_1 of $x_i = (\alpha_i t_2)(\alpha_i t)^2$.

If we completely polarize

$$x_1 = 3t^2, \quad x_2 = 3t, \quad x_3 = t^3 - 1,$$

we have

$$x_1 = s_2, \quad x_2 = s_1, \quad x_3 = s_3 - 1.$$

The points of inflexion of this curve are given by $t^3-1=0$, and the stationary tangents are

$$x_1-x_2-x_3=0, \quad x_1-\omega x_2-\omega^2 x_3=0, \quad x_1-\omega^2 x_2-\omega x_3=0.$$

If in the above polarized equations we put $s_1=s_2=0$, $s_3=1$, we have $x_1=x_2=x_3=0$. We may partially polarize the equations, writing

$$x_1=\sigma_2+\sigma_1 t, \quad x_2=\sigma_1+t, \quad x_3=\sigma_2 t-1.$$

In general, the two points given by $t^2-\sigma_1 t+\sigma_2=0$ will in this way determine a line; but if the roots of this quadratic are the parameters of two points of inflexion, the line degenerates into a point. Thus if

$$t^2-\sigma_1 t+\sigma_2\equiv(t-\omega)(t-\omega^2)\equiv t^2+t+1,$$

we have

$$\begin{aligned} x_1&=1-t &&\equiv -1\,.\,(t-1),\\ x_2&=-1+t &&\equiv 1\,.\,(t-1),\\ x_3&=t-1 &&\equiv 1\,.\,(t-1).\end{aligned}$$

The point $(-1, 1, 1)$ is the meet of the stationary tangents

$$x_1-\omega x_2-\omega^2 x_3=0 \quad\text{and}\quad x_1-\omega^2 x_2-\omega x_3=0.$$

If we partially polarize the equation of the quartic

$$x_i=a_i t^4+4b_i t^3+6c_i t^2+4d_i t+e_i,$$

writing

$$x_i=a_i s_3 t+b_i(s_3+s_2 t)+c_i(s_2+s_1 t)+d_i(s_1+t)+e_i,$$

or

$$x_i=(a_i s_3+b_i s_2+c_i s_1+d_i)\left(t+\frac{b_i s_3+c_i s_2+d_i s_1+e_i}{a_i s_3+b_i s_2+c_i s_1+d_i}\right),$$

in general the three parameters (t_1, t_2, t_3) given by $t^3-s_1t^2+s_2t-s_3=0$ determine a line. If, however, they belong to a set of the I_4, of which t_4 is the fourth member, we know that $x_i=0$ if $t=t_4$. Hence the above expressions for x_i are divisible by $t-t_4$ and the line degenerates into a point. The points of inflexion of the osculant cubic at t_3 are given by t_1, t_2 and t_4, and if we partially polarize its equations and substitute t_1+t_2 for σ_1, and t_1t_2 for σ_2, we evidently have the same point as the above. But we have seen that the point obtained by the latter method is the meet of the stationary tangents at t_1 and t_2 of the osculant cubic at t_3.

Obviously the stationary tangents at t_2 and t_3 of the osculant cubic at t_1, and the stationary tangents at t_3 and t_1 of the osculant cubic at t_2, are on this point. The locus of this point is, then, the conic which Stahl* calls K; *i. e.*, the locus of meets of stationary tangents of osculant cubics. For a set of the I_4 of which α is a member we have the three equations

$$-\alpha = \frac{b_i s_3 + c_i s_2 + d_i s_1 + e_i}{a_i s_3 + b_i s_2 + c_i s_1 + d_i}.$$

From these equations we can express s_1, s_2 and s_3 in terms of α; and, substituting these values in

$$x_i = a_i s_3 + b_i s_2 + c_i s_1 + d_i,$$

we have the equations of K with α as the parameter in essentially the same form as given by Stahl.

Stahl has shown that if t_1, t_2, t_3 and t_4 form a set of the I_4, any two of the osculant cubics at these points have a common stationary tangent. There are thus in all six lines arranged as in the following diagram (Fig. 1).

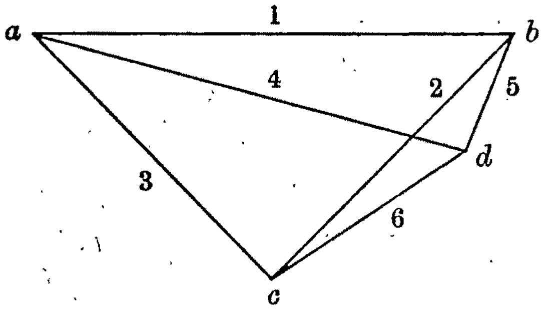

FIG. 1.

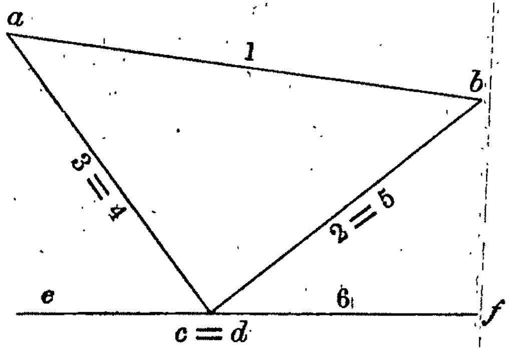

FIG. 2.

The four points of K determined by a set of the I_4 are a, b, c and d. If the lines 1, 2 and 3 are the stationary tangents of the osculant at t_4, d is the point of K given by $\alpha = t_4$.

§ 3. *The Covariant Curves g_2 and g_3.*

If we assume that the parameters of two of the points of inflexion of a plane rational quartic are 0 and ∞, we may, in general, write its equations

$$x_1 = a_1 t^4 + b_1 t^3, \quad x_2 = d_2 t + e_2, \quad x_3 = b_3 t^3 + c_3 t^2 + d_3 t.$$

We shall call the curve Q.

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* *Crelle*, Vol. CI, p. 306.

If $b_3 = 0$ or $d_3 = 0$, the curve is cuspidal; if $b_1 = 0$ or $d_2 = 0$, it has a point of undulation. The first osculant at the point τ may be written

$$x_1 = (4\,a_1\tau + b_1)\,t^3 + 3\,b_1\tau\,t^2, \quad x_2 = 3\,d_2\,t + d_2\tau + 4\,e_2,$$
$$x_3 = b_3\,t^3 + (3\,b_3\tau + 2\,c_3)\,t^2 + (2\,c_3\tau + 3\,d_3)\,t + d_3\tau.$$

If $\tau = 0$, this becomes

$$x_1 = b_1 t^3, \quad x_2 = 3\,d_2\,t + 4\,e_2, \quad x_3 = b_3\,t^3 + 2\,c_3\,t^2 + 3\,d_3\,t,$$

whence it is evident that the osculant cubic at a point of inflexion has itself a point of inflexion there, and that the osculant cubic at a point of undulation is the undulation tangent counted three times (since for each value of $\frac{x_2}{x_3}$ there are three values of t).

If $d_3 = 0$, all the cubic osculants touch the cusp tangent, $x_1 = 0$, at the cusp. The inflexion tangents ($x_1 = 0$ and $x_2 = 0$) meet Q again at the points given by $a_1 t + b_1 = 0$ and $d_2 t + e_2 = 0$; the osculant cubic at the point $\tau = \frac{-4\,e_2}{d_2}$ touches $x_1 = 0$ at the meet of these inflexion tangents, while the osculant at $\tau = -\frac{b_1}{4\,a_1}$ touches $x_2 = 0$ at the same point.

The equations of the osculant in lines are

$$\begin{aligned}\xi_1 = {} & 6\,b_3\,d_2\,t^3 + 3\,(b_3\,(d_2\tau + 4\,e_2) + d_2\,(3\,b_3\tau + 2\,c_3))\,t^2 \\ & + 2\,(d_2\tau + 4\,e_2)\,(3\,b_3\tau + 2\,c_3)\,t + 2\,(c_3\,d_2\,\tau^2 + 4\,c_3\,e_2\,\tau + 6\,d_3\,e_2), \\ \xi_2 = {} & t\,[2\,(6\,a_1\,b_3\,\tau^2 + 4\,a_1\,c_3\,\tau + b_1\,c_3)\,t^3 + 2\,(4\,a_1\tau + b_1)\,(2\,c_3\tau + 3\,d_3)\,t^2 \\ & + 3\,\tau\,(d_3\,(4\,a_1\tau + b_1) + b_1\,(2\,c_3\tau + 3\,d_3))\,t + 6\,b_1\,d_3\,\tau^3], \\ \xi_3 = {} & -3\,t\,[2\,d_2\,(4\,a_1\tau + b_1)\,t^2 + ((4\,a_1\tau + b_1)\,(d_2\tau + 4\,e_2) + 3\,b_1\,d_2\,\tau)\,t \\ & + 2\,b_1\,\tau\,(d_2\tau + 4\,e_2)].\end{aligned}$$

If $\tau = t$ in these equations, we have the line equations of Q; viz.,

$$\begin{aligned}\xi_1 &= 12\,[2\,b_3\,d_2\,t^3 + (3\,b_3\,e_2 + c_3\,d_2)\,t^2 + 2\,c_3\,e_2\,t + d_3\,e_2], \\ \xi_2 &= 12\,t^3\,[a_1\,b_3\,t^3 + 2\,a_1\,c_3\,t^2 + (3\,a_1\,d_3 + b_1\,c_3)\,t + 2\,b_1\,d_3], \\ \xi_3 &= -12\,t^2\,[3\,a_1\,d_2\,t^2 + 2\,(2\,a_1\,e_2 + b_1\,d_2)\,t + 3\,b_1\,e_2].\end{aligned}$$

The four points in which Q meets the line $(x\,\xi) = 0$* are given by

$$a_1\,\xi_1\,t^4 + (b_1\,\xi_1 + b_3\,\xi_3)\,t^3 + c_3\,\xi_2\,t^2 + (d_2\,\xi_2 + d_3\,\xi_3)\,t + e_2\,\xi_2 = 0.$$

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* $(x\,\xi) \equiv x_1\,\xi_1 + x_2\,\xi_2 + x_3\,\xi_3$.

The roots of this equation form a self-apolar set if the line touches the conic

$$g_2 \equiv (c_3^2 - 3\,b_3 d_3)\,\xi_3^2 - 3\,b_3 d_2 \xi_2 \xi_3 - 3\,b_1 d_3 \xi_1 \xi_3 + 3\,(4\,a_1 e_2 - b_1 d_2)\,\xi_1 \xi_2 = 0,$$

or, in points,

$$3(b_3 d_2 x_1 + b_1 d_3 x_2 + (4\,a_1 e_2 - b_1 d_2)\,x_3)^2 + 4(c_3^2(4\,a_1 e_2 - b_1 d_2) - 12\,a_1 b_3 d_3 e_2)\,x_1 x_2 = 0.$$

If $d_3 = 0$, Q has a cusp at $(0, 1, 0)$; g_2 then touches the cusp tangent at the cusp. The points where g_2 meets Q are given by

$$\begin{aligned} &3\,(a_1 b_3 d_2 t^4 + 4\,a_1 b_3 e_2 t^3 - 4\,a_1 d_3 e_2 t - b_1 d_3 e_2)^2 + 36\,a_1 b_3 d_3 e_2 (4\,a_1 e_2 - b_1 d_2)\,t^4 \\ &\quad + 6\,c_3 (4\,a_1 e_2 - b_1 d_2)(a_1 b_3 d_2 t^4 + 4\,a_1 b_3 e_2 t^3 + 4\,a_1 d_3 e_2 t + b_1 d_3 e_2)\,t^2 \\ &\quad + c_3^2 (4\,a_1 e_2 - b_1 d_2)(4\,a_1 t + b_1)(d_2 t + 4\,e_2)\,t^3 = 0. \end{aligned}$$

The discriminant of g_2 is

$$\frac{9}{4}(4\,a_1 e_2 - b_1 d_2)(12\,a_1 b_3 d_3 e_2 - c_3^2(4\,a_1 e_2 - b_1 d_2)).$$

If $(4\,a_1 e_2 - b_1 d_2) = 0$, g_2 becomes

$$\xi_3((c_3^2 - 3\,b_3 d_3)\,\xi_3 - 3\,b_3 d_2 \xi_2 - 3\,b_1 d_3 \xi_1) = 0$$

and three stationary tangents of Q meet at each of the points given by this equation. Four stationary tangents can not be on the same point, since then there would be eight tangents to Q from this point.

If $d_3 = 0$, g_2 meets Q at the points

$$\begin{aligned} &3\,a_1^2 b_3^2 (d_2 t + 4\,e_2)^2 t^6 + 6\,a_1 b_3 c_3 (4\,a_1 e_2 - b_1 d_2)(d_2 t + 4\,e_2)\,t^5 \\ &\qquad + c_3^2 (4\,a_1 e_2 - b_1 d_2)(4\,a_1 t + b_1)(d_2 t + 4\,e_2)\,t^3 = 0; \end{aligned}$$

i. e., g_2 touches the cusp tangent at the cusp and one of the five other points of intersection is given by $d_2 t + 4\,e_2 = 0$.

If also $b_3 = 0$, and $4\,a_1 e_2 - b_1 d_2 = 0$, the quartic is tricuspidal: g_2 is then $c_3^2 \xi_3^2 = 0$, and the equation giving its intersections with Q vanishes identically.

Obviously g_2 touches the stationary tangents of Q: it touches $x_1 = 0$ and $x_2 = 0$ at the points $(0,\ 4\,a_1 e_2 - b_1 d_2,\ -b_1 d_3)$ and $(4\,a_1 e_2 - b_1 d_2,\ 0,\ -b_3 d_2)$ respectively.

Knowing a stationary tangent of Q, we can put its points in one-to-one correspondence with the lines of g_2. Thus the line $x_2 = 0$ touches the osculant at the point given by $t = \infty$, viz.:

$$x_1 = 4\,a_1 \tau + b_1, \quad x_2 = 0, \quad x_3 = b_3.$$

The equation of this point is $(4\,a_1 \tau + b_1)\,\xi_1 + b_3 \xi_3 = 0$, the correspondence between ξ_1/ξ_3 and τ being one-to-one.

From this point there is one tangent, other than $x_3 = 0$, to g_2. If in the line equation of g_2 we substitute $\xi_1 = -\frac{b_3 \xi_3}{4 a_1 \tau + b_1}$, we obtain parametric equations of the conic in the form

$$\begin{aligned}\xi_1 &= -12 a_1 b_3^2 (d_2 \tau + e_2),\\ \xi_2 &= (c_3^2 (4 a_1 \tau + b_1) - 12 a_1 b_3 d_3 \tau)(4 a_1 \tau + b_1),\\ \xi_3 &= 12 a_1 b_3 (d_2 \tau + e_2)(4 a_1 \tau + b_1).\end{aligned}$$

If $b_3 = 0$, all the first osculants touch $x_2 = 0$ at the cusp and these equations can not be used. The osculant at $\tau = -\frac{e_2}{d_2}$ is

$$x_1 = -(4 a_1 e_2 - b_1 d_2) t^3 - 3 b_1 e_2 t^2, \quad x_2 = 3 d_2 (d_2 t + e_2),$$
$$x_3 = b_3 d_2 t^3 - (3 b_3 e_2 - 2 c_3 d_2) t^2 - (2 c_3 e_2 - 3 d_2 d_3) t - d_3 e_2.$$

If $t = \infty$,

$$x_1 = -(4 a_1 e_2 - b_1 d_2), \quad x_2 = 0, \quad x_3 = b_3 d_2.$$

Hence the osculant cubic of a quartic, at the point at which a tangent at a point of inflexion meets it again, touches that tangent at the point at which g_2 touches it. Since the above correspondence between τ and ξ_1/ξ_3 is one-to-one, it is evident that, in general, there is no pair of points on Q such that the osculant cubics at these points coincide.

There are eight tangents common to g_2 and an osculant cubic. Of these six are the stationary tangents of Q. The two variable tangents are given, on the osculant, by

$$(d_2 \tau + e_2) \rho_\tau t^2 + 2 [3 a_1 b_3 \tau^3 (d_2 \tau + 4 e_2) + c_3 (4 a_1 \tau + b_1)(d_2 \tau + 4 e_2) \tau$$
$$+ 3 d_3 e_2 (4 a_1 \tau + b_1)] t - \tau (a_1 \tau + b_1) \sigma_\tau = 0,$$

where

$$\begin{aligned}\rho_\tau &= 2 (6 a_1 b_3 \tau^2 + 4 a_1 c_3 \tau + b_1 c_3),\\ \sigma_\tau &= -2 (c_3 d_2 \tau^2 + 4 c_3 e_2 \tau + 6 d_3 e_2).\end{aligned}$$

Two of the eight tangents will coincide if a variable tangent coincides with one of the stationary tangents, or if the two variable tangents coincide. Thus $t = 0$ gives $x_1 = 0$. One of the variable tangents is given by $t = 0$ if:

(a) $\tau = 0$, *i. e.*, $x_1 = 0$ is a stationary tangent of the osculant;

(b) $a_1 \tau + b_1 = 0$, when the osculant touches g_3 where $x_1 = 0$ touches it;

(c) $\sigma_\tau = 0$, in which case, as is evident from its line equations, the osculant is cuspidal and has only six proper tangents in common with g_3.

The two variable tangents coincide if

$$\tau(a_1\tau+b_1)(d_2\tau+e_2)\rho_\tau\sigma_\tau+[3a_1b_3\tau^3(d_2\tau+4e_2)$$
$$+c_3(4a_1\tau+b_1)(d_2\tau+4e_2)\tau+3d_3e_2(4a_1\tau+b_1)]^2=0.$$

This equation is of the eighth order in τ and on expansion proves to be identical with that giving the intersections of Q with g_2.

The osculant at such a point has a node at the point of contact, the nodal tangents being the tangent to Q and the tangent to g_2. It follows that the node of an osculant cubic of a tricuspidal quartic is at the point of contact and that one of the nodal tangents passes through the meet of the cusp tangents.

The points of contact of the two tangents to the cubic

$$x_i=a_it^3+b_it^2+c_it+d_i$$

from the point τ on it are given by the equation

$$[|abc|\tau+|abd|]t^2+2[|abd|\tau+|acd|]t+|acd|\tau+|bcd|=0.$$

If we apply this formula to the cubic osculant of Q at τ, *we find that the two tangents to the osculant from the point of contact coincide with the two variable common tangents of g_2 and the osculant.*

The equations of the conic K are

$$x_1=-3b_1^2\sigma_\alpha,\qquad x_2=3d_2^2\rho_\alpha,$$
$$x_3=3(d_2d_3\rho_\alpha-b_1b_3\sigma_\alpha)+144a_1b_3d_3e_2\alpha-4c_3^2(4a_1\alpha+b_1)(d_2\alpha+4e_2).$$

If $b_1=0$, $x_1=0$ is an undulation tangent of Q, and K is degenerate. In fact, it is evident that an undulation tangent of Q is a stationary tangent of each cubic osculant, and that, therefore, every point on it is a point of K.

If $d_3=0$, K becomes

$$x_1=6b_1^2c_3(d_2\alpha+4e_2)\alpha,\qquad x_2=3d_2^2\rho_\alpha,$$
$$x_3=2(d_2\alpha+4e_2)(3b_1b_3c_3\alpha-2c_3^2(4a_1\alpha+b_1)).$$

K is on the cusp, but does not touch the cusp tangent.

In general, $x_1=0$ meets K when $\rho_\alpha=0$. The equation of the two points given by these values of α is

$$3d_3^2\xi_2^2+2(3d_2d_3-4c_3e_2)\xi_2\xi_3+3d_3^2\xi_3^2=0.$$

The locus of lines such that the parameters of the points in which they cut Q form harmonic sets is

$$g_3 \equiv \begin{vmatrix} a_1\xi_1, & \dfrac{b_1\xi_1 + b_3\xi_3}{4}, & \dfrac{c_3}{6} \\ \dfrac{b_1\xi_1 + b_3\xi_3}{4}, & \dfrac{c_3}{6}, & \dfrac{d_2\xi_2 + d_3\xi_3}{4} \\ \dfrac{c_3}{6}, & \dfrac{d_2\xi_2 + d_3\xi_3}{4}, & e_2\xi_2 \end{vmatrix} = 0,$$

or

$$c_3\xi_3(72\,a_1e_2\xi_1\xi_2 + 9\,(b_1\xi_1 + b_3\xi_3)(d_2\xi_2 + d_3\xi_3) - 2\,c_3^2\xi_3^2) \\ - 27\,a_1\xi_1(d_2\xi_2 + d_3\xi_3)^2 - 27\,e_2\xi_2(b_1\xi_1 + b_3\xi_3)^2 = 0.$$

Evidently g_3 touches Q at the points of inflexion.

If the roots of the equation $a_0t^3 + 3\,a_1t^2 + 3\,a_2t + a_3 = 0$, are equal,

$$a_0t + a_1 = 0, \quad a_1t + a_2 = 0 \text{ and } a_2t + a_3 = 0.$$

A cubic osculant meets the line $(x\xi) = 0$ when

$$((4\,a_1\tau + b_1)\,\xi_1 + b_3\xi_3)\,t^3 + (3\,b_1\tau\xi_1 + (3\,b_3\tau + 2\,c_3)\,\xi_3)\,t^2 \\ + (3\,d_2\xi_2 + (2\,c_3\tau + 3\,d_3)\,\xi_3)\,t + (d_2\tau + 4e_2)\,\xi_2 + d_3\tau\xi_3 = 0.$$

If the line is a stationary tangent of the osculant, we have from the first of the above conditions

$$3\,((4\,a_1\tau + b_1)\,\xi_1 + b_3\xi_3)\,t + 3\,b_1\tau\xi_1 + (3\,b_3\tau + 2\,c_3)\,\xi_3 = 0,$$

or

$$12\,a_1\xi_1\tau t + 3\,(b_1\xi_1 + b_3\xi_3)(\tau + t) + 2\,c_3\xi_3 = 0.$$

In like manner the other conditions give

$$3\,(b_1\xi_1 + b_3\xi_3)\,\tau t + 2\,c_3\xi_3(\tau + t) + 3\,(d_2\xi_2 + d_3\xi_3) = 0$$

and

$$2\,c_3\xi_3\tau t + 3\,(d_2\xi_2 + d_3\xi_3)(\tau + t) + 12\,e_2\xi_2 = 0.$$

Eliminating τt and $\tau + t$, we find that the envelope of the stationary tangents of osculant cubics is g_3.*

In general, a set of the I_4 of Q determines four points on K. Let us consider what happens if the set is one to which the parameter of a point of inflexion of Q (say $t = 0$) belongs. One of the stationary tangents of the cubic osculant at this point is $x_1 = 0$, given by $t = 0$; hence 0 is to be counted twice as a member of this set. We may call the two other members α and β. The osculant at $t = 0$

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* Morley: *Trans. Am. Math. Soc.*, Vol. VIII, p. 17. Stahl: *Crelle*, Vol. CI, p. 302.

has three distinct stationary tangents given by $t=0$, $t=\alpha$ and $t=\beta$; the osculant at $t=\alpha$ and the osculant at $t=\beta$ have each two stationary tangents given by $t=0$, and one given by $t=\beta$ and $t=\alpha$ respectively; *i. e.*, these latter osculants are cuspidal. The points c and d of Fig. 1 (page 216) coincide as in Fig. 2. The stationary tangents of the osculant at $t=0$ are ab, bc and ca; bc and ca are cuspidal tangents of the osculants at $t=\alpha$ and $t=\beta$ respectively, while these two osculants have a common stationary tangent, ef. The line $x_1=0$ is not a proper tangent of the cuspidal osculants, but merely passes through the cusps.

In general, from a point of K there are three distinct tangents to g_3. At each of the points a and b two of these tangents to g_3 coincide. Hence a and b are on g_3, bc and ca being tangents of g_3.

Since, in general, g_3 is of the sixth order, it intersects K in twelve points which are thus accounted for.

Of the thirty-six intersections of the two sextics, g_3 and the six stationary tangents of Q, twelve are on the conic K; there is, therefore, a curve of the fourth order touching g_3 at the six points of inflexion of Q and passing through the other twelve intersections.

The line ef, being the limiting position of the chord cd of K, touches K. *In this way we can account for the six common tangents of g_3 and K.*

§ 4. *The Nodal Locus of Osculant Cubics.*

Stahl has shown that the envelope of the lines joining the points of inflexion of osculant cubics of Q is a conic. The locus of the nodes of these cubics is evidently a rational curve, the equations of which we will now find.

In general, the equations of a rational cubic may by a change of reference triangle be put in the form

$$x_1=A_1t^3+B_1t^2,\quad x_2=C_2t+D_2,\quad x_3=A_3t^3+B_3t^2+C_3t+D_3.$$

The line $(x\xi)=0$ meets this curve at the points

$$(\xi_1A_1+\xi_3A_3)t^3+(\xi_1B_1+\xi_3B_3)t^2+(\xi_2C_2+\xi_3C_3)t+(\xi_2D_2+\xi_3D_3)=0.$$

The quadratic $m_0t^2+m_1t+m_2=0$ is a factor of this if

$$\frac{m_0(\xi_2C_2+\xi_3C_3)-m_2(\xi_1A_1+\xi_3A_3)}{m_0(\xi_1B_1+\xi_3B_3)-m_1(\xi_1A_1+\xi_3A_3)}=\frac{m_1}{m_0},$$

and

$$\frac{m_0(\xi_2D_2+\xi_3D_3)}{m_0(\xi_1B_1+\xi_3B_3)-m_1(\xi_1A_1+\xi_3A_3)}=\frac{m_2}{m_0}.$$

In general, these two equations determine the ratios $\xi_1:\xi_2:\xi_3$, and thus a definite line. If, however, $m_0 t^2 + m_1 t + m_2 = 0$ gives the parameters of the double point of the cubic, the two equations are equivalent and give merely the equation of the double point. For this cubic the parameters of the points of inflexion are given by

$$C_2 R t^3 + 3 D_2 R t^2 - 3 A_1 S t - B_1 S = 0,$$

and those of the double point by

$$(D_2^2 R + A_1 C_2 S) R t^2 - (A_1 D_2 - B_1 C_2) R S t + (B_1 D_2 R + A_1^2 S) S = 0,$$

where

$$R = A_1 B_3 - A_3 B_1 \text{ and } S = C_2 D_3 - C_3 D_2.$$

The cubic is cuspidal if

$$R S [(A_1 D_2 - B_1 C_2)^2 R S - 4 (D_2^2 R + A_1 C_2 S)(B_1 D_2 R + A_1^2 S)] = 0.$$

Using the second of the conditions given above, we find, after removal of the common factor $R D_2$, for the coordinates of the double point:

$$x_1 = -(B_1 D_2 R + A_1^2 S)^2 S, \quad x_2 = (D_2^2 R + A_1 C_2 S)^2 R,$$
$$x_3 = D_2^3 D_3 R^3 - D_2 (B_1 B_3 D_2 - 2 A_1 C_2 D_3) R^2 S$$
$$+ A_1 (A_1 C_2 C_3 - 2 A_3 B_1 D_2) R S^2 - A_1^3 A_3 S^3 - (A_1 D_2 + B_1 C_2) R^2 S^2.$$

If $D_3 = 0$, we have

$$x_1 = C_3 D_2^3 (B_1 R - A_1^2 C_3)^2, \quad x_2 = D_2^2 (D_2 R - A_1 C_2 C_3)^2 R,$$
$$x_3 = D_2^2 [(B_1 B_3 C_3 D_2 - A_1 C_3^2 D_2 - B_1 C_2 C_3^2) R^2$$
$$+ A_1 C_3^2 (A_1 C_2 C_3 - 2 A_3 B_1 D_2) R + A_1^3 A_3 C_3^3 D_2].$$

If also $A_3 = 0$,

$$x_1 = A_1^2 D_2^2 . C_3 D_2 (A_1 C_3 - B_1 B_3)^2, \quad x_2 = A_1^2 D_2^2 . A_1 B_3 (B_3 D_2 - C_2 C_3)^2,$$
$$x_3 = -A_1^2 D_2^2 . B_3 C_3 (A_1 C_3 - B_1 B_3)(B_3 D_2 - C_2 C_3).$$

Using the formula of page 214, we have, for the I_4 of Q,

$$b_1 t^2 (c_3 d_2 t^2 + 4 c_3 e_2 t + 6 d_3 e_2) + d_2 s_4 (6 a_1 b_3 t^2 + 4 a_1 c_3 t + b_1 c_3) = 0,$$

or, more briefly,

$$t^2 \sigma_t + k \rho_t = 0.$$

If $d_3 = 0$, Q has a cusp and the I_4 becomes

$$b_1 c_3 t^3 (d_2 t + 4 e_2) + d_2 s_4 (6 a_1 b_3 t^2 + 4 a_1 c_3 t + b_1 c_3) = 0.$$

If also $b_3 = 0$, Q is bicuspidal and I_4 becomes

$$c_3 [b_1 t^3 (d_2 t + 4 e_2) + d_2 s_4 (4 a_1 t + b_1)] = 0.$$

If $a_1 = d_2 = 1$, $b_1 = 4$ and $e_2 = 1$, so that Q is tricuspidal, I_4 reduces to

$$t^3 (t + 4) + s_4 (t + 1) = 0.$$

In this case the four points in which a line on the meet of the cusp tangents cuts Q form a set of I_4.

It is known that if a conic touch the four bitangents of a rational quartic, the four other common tangents touch the quartic at points which form a set of I_4.* Hence we may infer that *if a conic touch the bitangent and the joins of the three cusps of a tricuspidal quartic, the four other common tangents touch the quartic at points on a line on the meet of the cusp tangents.*

We may verify this inference as follows. For the quartic

$$x_1 = t^4 + 4t^3, \quad x_2 = t + 1, \quad x_3 = t^2,$$

the cusps are at the points

$$(0, 1, 0), \quad (1, 0, 0), \quad (16, 1, -4).$$

Their joins are

$$x_3 = 0, \quad 4x_2 + x_3 = 0, \quad x_1 + 4x_3 = 0.$$

The bitangent is $x_1 + 16x_2 + 12x_3 = 0$, given by $t^2 + 2t + 4 = 0$. These four lines are the base-lines of the range of conics

$$\xi_2(32\xi_1 + \xi_2 - 4\xi_3) + \lambda\xi_1(8\xi_1 + \xi_2 - 2\xi_3) = 0.$$

The line equations of the quartic are

$$\xi_1 = 1, \quad \xi_2 = 2t^3, \quad \xi_3 = -3t(t + 2).$$

Substituting, we find for the common tangents

$$2t^3(t^3 + 6t^2 + 12t + 16) + \lambda(t^3 + 3t^2 + 6t + 4) = 0,$$

or, removing the common factor $t^2 + 2t + 4$,

$$2t^3(t + 4) + \lambda(t + 1) = 0,$$

which is I_4.

Hence, if any parabola be inscribed in the triangle of which the cusps of a deltoid are the vertices, the four common tangents, other than the line at infinity, will touch the deltoid at points which are on a line on its center.

Identifying the osculant cubic of Q with the cubic of page 222, we have

$$\begin{aligned} A_1 &= 4a_1\tau + b_1, & B_1 &= 3b_1\tau, \\ C_2 &= 3d_2, & D_2 &= d_2\tau + 4e_2, \\ A_3 &= b_3, & B_3 &= 3b_3\tau + 2c_3, \\ C_3 &= 2c_3\tau + 3d_3, & D_3 &= d_3\tau, \end{aligned}$$

$$R = 2(6a_1b_3\tau^2 + 4a_1c_3\tau + b_1c_3) \equiv \rho_\tau,$$
$$S = -2(c_3d_2\tau^2 + 4c_3e_2\tau + 6d_3e_2) \equiv \sigma_\tau.$$

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* Stahl: *Crelle*, Vol. CI, p. 302.

Thus we have, for the double point of the osculant,

$$\begin{aligned} x_1 = 8\,[3\,b_1\tau\,(d_2\tau + 4\,e_2)\,(6\,a_1 b_3\tau^2 + 4\,a_1 c_3\tau + b_1 c_3) \\ + (4\,a_1\tau + b_1)^2\,(c_3 d_2\tau^2 + 4\,c_3 e_2\tau + 6\,d_3 e_2)]^2\,[c_3 d_2\tau^2 + 4\,c_3 e_2\tau + 6\,d_3 e_2], \end{aligned}$$

and corresponding expressions for x_2 and x_3.

Varying τ, we have the equations of the nodal locus of osculant cubics. It is a rational curve of the tenth order. The stationary tangent, $x_1 = 0$, of Q cuts it at the points given by $\sigma_\tau = 0$ and touches it at each of the four points given by the other factor in the above expression for x_1; $\sigma_\tau = 0$ gives the two points of Q which form a set of I_4 with $t^2 = 0$; also, when $\sigma_\tau = 0$ the cubic is cuspidal. Since all the osculant cubics touch $x_1 = 0$, this line must be a nodal tangent when a node is on it. The decimic is evidently symmetric as to the six stationary tangents of Q. Hence each stationary tangent cuts it at two points which are cusps of osculants, and touches it at four points at each of which this tangent is a nodal tangent of an osculant. Since the condition that the osculant cubic be cuspidal is of the twelfth degree in τ, this accounts for all the cuspidal osculants. The four points given by the first factor in the expression for x_1 are not, in general, cusps of the decimic, since it can not have twenty-four cusps.

If we let $t = \tau$ in the equation giving the parameters of the node of an osculant, it is of the eighth order in τ, and gives the equation of the eight points of Q at which, in general, the node of the osculant is at the point of contact. It may be easily verified that this equation coincides with that giving the intersections of Q with g_2.

At least eight of the forty intersections of Q with the decimic are at these points. If $d_3 = 0$, Q has a cusp given by $t = 0$. Then $(d_2\tau + 4\,e_2)^2 = 0$ is a common factor of the equations of the decimic, which reduces to an octavic; the osculant cubic at the point given by $d_2\tau + 4e_2 = 0$ degenerates; $d_2 t + 4e_2 = 0$ gives that value of t which with $t^3 = 0$ forms a set of I_4.

When $d_3 = 0$, the osculant at $\tau = \dfrac{-4\,e_2}{d_2}$ is

$$x_1 = -\,[(16\,a_1 e_2 - b_1 d_2)\,t + 12\,b_1 e_2]\,t^2, \quad x_2 = 3\,d_2^2\,t,$$
$$x_3 = b_3 d_2 t^3 - 2\,(6\,b_3 e_2 - c_3 d_2)\,t^2 - 8\,c_3 e_2 t;$$

or $t = 0$ and

$$x_1 = -\,[(16\,a_1 e_2 - b_1 d_2)\,t + 12\,b_1 e_2]\,t, \quad x_2 = 3\,d_2^2,$$
$$x_3 = b_3 d_2 t^2 - 2\,(6\,b_3 e_2 - c_3 d_2)\,t - 8\,c_3 e_2.$$

This conic touches $x_2 = 0$, and consequently the three other stationary tangents of Q. If in these equations $t = 0$,

$$x_1 = 0, \quad x_2 = 3\,d_2^2, \quad x_3 = -8\,c_3 e_2.$$

The point of contact, given by $t = \frac{-4\,e_2}{d_2}$, is

$$x_1 = -64\,e_2^2 (4\,a_1 e_2 - b_1 d_2), \quad x_2 = 3\,d_2^4, \quad x_3 = 16\,d_2 e_2 (4\,b_3 e_2 - c_3 d_2).$$

The join of these points is

$$3\,d_2^3 (8\,b_3 e_2 - c_3 d_2)\,x_1 + 64\,c_3 e_2^2 (4\,a_1 e_2 - b_1 d_2)\,x_2 \\ + 24\,d_2^2 e_2 (4\,a_1 e_2 - b_1 d_2)\,x_3 = 0,$$

which proves to be a line of g_2.

We have already shown that, when $d_3 = 0$, $d_2 t + 4\,e_2 = 0$ gives an intersection of g_2 with Q. Hence, there is only one tangent to g_2 from this point, and consequently there can be only one tangent to the osculant from it.

It is now evident that the conic and any line on the point given by $t = 0$ *on it form a curve of the third order which fulfills all the conditions necessary for an osculant cubic.*

Returning to the decimic, if also $b_3 = 0$, Q has a second cusp at $t = \infty$. Then $(4\,a_1 \tau + b_1)^2$ is a common factor of the equations of the octavic which reduces to a sextic; the osculant cubic for this value of τ degenerates; $4\,a_1 \tau + b_1 = 0$ forms a set of I_4 with $t^3 = \infty$.

The sextic then is

$$x_1 = \tau^3 (d_2 \tau + 4\,e_2)(2\,a_1 \tau - b_1)^2, \quad x_2 = (4\,a_1 \tau + b_1)(d_2 \tau - 2\,e_2)^2, \\ x_3 = 2\,c_3^2 \tau^2 (2\,a_1 \tau - b_1)(d_2 \tau - 2\,e_2).$$

If $a_1 = d_2 = c_3 = e_2 = 1$ and $b_1 = 4$, this becomes $4\,(\tau - 2)^2 = 0$ and

$$x_1 = \tau^3 (\tau + 4), \quad x_2 = \tau + 1, \quad x_3 = \tau^2,$$

which now coincides with Q.

The equations of the osculant for this curve are

$$x_1 = 4\,(\tau + 1)\,t^3 + 12\,\tau t^2, \quad x_2 = 3\,t + (\tau + 4), \quad x_3 = 2\,t^2 + 2\,\tau t.$$

On page 223 we have the equation giving the parameters of the double point of a cubic given in this form. Here

$$A_1 = 4\,(\tau + 1), \qquad B_1 = 12\,\tau,$$
$$C_2 = 3, \qquad D_2 = \tau + 4,$$
$$R = 8\,(\tau + 1), \qquad S = -2\,\tau (\tau + 4).$$

Hence we find for the parameters of the double point, after removal of the factor

$$64(\tau+1)(\tau+4)(\tau-2),$$

for which values of τ we have seen that the double point is indeterminate,[*]

$$(t-\tau)(2(\tau+1)t+\tau(\tau+4))=0,$$

thus verifying the statement that on a tricuspidal quartic the node of an osculant cubic is at the point of contact.

We have already (page 211) identified the curve which corresponds to the factor $(\tau-2)^2$ on page 226.

Taking I_4 in the form $t^2\sigma_t+k\rho_t=0$, the two values of k which make this equation self-apolar are given by

$$c_3^2 k(b_1 d_2-4a_1 e_2)+3(d_3 e_2+a_1 b_3 k)^2=0.\dagger$$

Eliminating k, we have for the two self-apolar sets of I_4, after removal of the factor c_3^2,

$$12(a_1 b_3 d_2 t^4+4a_1 b_3 e_2 t^3-4a_1 d_3 e_2 t-b_1 d_3 e_2)^2-(4a_1 e_2-b_1 d_2)t^2\rho_t\sigma_t=0.$$

This equation is the same as that giving the intersections of g_2 with Q.

Hence the two self-apolar sets of I_4 give on Q the intersections of Q with g_2; cubic osculants at these points have double points at the point of contact.

It follows that every set of I_4 is self-apolar when Q has three cusps.

§ 5. *An Interpretation of Syzygies.*

The cubic osculant of

$$x_1=t^4+4t^3,\quad x_2=t+1,\quad x_3=t^2$$

being

$$x_1=4t^3(\tau+1)+12\tau t^2,\quad x_2=3t+\tau+4,\quad x_3=2t^2+2\tau t,$$

the elimination of t gives

$$(2x_1x_2^2-6x_2x_3^2-x_3^3)\tau^3-3x_3(x_1x_2+8x_2x_3+x_3^2)\tau^2$$
$$+3x_3(2x_1x_2+x_1x_3+2x_3^2)\tau-(x_1^2x_2-3x_1x_3^2-8x_3^3)=0.$$

In lines, the equation of the quartic is

$$\xi_1=1,\quad \xi_2=2t^3,\quad \xi_3=-3t^2-6t.$$

The points of contact of the tangents from the point $(x\xi)=0$ are given by

$$2x_2t^3-3x_3t^2-6x_3t+x_1=0.$$

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[*] Since $\tau^3(\tau+4)$, $\tau+1$ and $(\tau+2)^2(\tau-2)$ are the sets of I_4 to which the parameters of the cusps belong.

[†] This equation can not be used if $\sigma_2=0$, since then we can not write the I_4 in the form here used.

The Hessian of this equation is, writing τ for t,

$$H \equiv -(x_3(4x_2+x_3)\tau^2 - 2(x_1x_2 - x_3^2)\tau + x_3(x_1 + 4x_3)) = 0.$$

Since the conic osculant of a point cubic touches the stationary tangents, the conic osculant of a line cubic is on the cusps.

The Hessian of a cubic has equal roots when the cubic has two equal roots, and vanishes identically when the cubic has three equal roots.

Therefore H touches the cubic at the point τ and is on the three cusps; *i. e.*, it is the point equation of the conic line osculant. If $\tau = 0$, H is the line osculant at the cusp $(0, 1, 0)$ and degenerates into the joins of this cusp with the other cusps, $(1, 0, 0)$ and $(16, 1, -4)$.*

The coefficient of τ equated to zero gives that conic on the three cusps which touches the cusp tangents $x_1 = 0$ and $x_2 = 0$, given by $\tau = 0$ and $\tau = \infty$.

In general, for the curve $x_i = (\alpha_i t)^n$ *the conic osculant is* $x_i = (\alpha_i t)^2(\alpha_i\tau)^{n-2}$, *and the process of finding the envelope of* $((\alpha_i t)^3(\alpha_i\tau)^{n-3}\xi_i) = 0$ *is the same as that of finding the Hessian of* $((\alpha_i t)^n\xi_i) = 0$.†

The cubic covariant of $2x_2\tau^3 - 3x_3\tau^2 - 6x_3\tau + x_1 = 0$ is

$$\begin{aligned}J \equiv 2\,[&(2x_1x_2^2 - 6x_2x_3^2 - x_3^3)\tau^3 - 3x_3(x_1x_2 + 8x_2x_3 + x_3^2)\tau^2 \\ &+ 3x_3(2x_1x_2 + x_1x_3 + 2x_3^2)\tau - (x_1^2x_2 - 3x_1x_3^2 - 8x_3^3)] = 0;\end{aligned}$$

i. e., $J = 0$ *is the point cubic osculant of the curve.*

Let U be the cubic $2x_2t^3 - 3x_3t^2 - 6x_3t + x_1$;

D, its discriminant;

H, its Hessian;

J, its cubic covariant;

so that

$U = 0$ is a tangent of the cubic;

$D = 0$ is the point equation;

$H = 0$ is the point equation of the conic line osculant;

$J = 0$ is the point equation of the cubic point osculant.

Then we know there exists a syzygy ‡

$$J^2 = DU^2 - 4H^3.$$

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* Dually the conic osculant of a point cubic at a point of inflexion τ, degenerates into the points where the stationary tangent at τ meets the other stationary tangents.

† Thus for Q, when $t = 0$, $H \equiv 3d_2^2\xi_2^2 + 2(3d_2d_3 - 4c_2e_2)\xi_2\xi_3 + 3d_3^2\xi_3^2 = 0$; *i. e.*, H is the points where $x_1 = 0$ meets K (page 230). H touches g_3 three times, since these points are on g_3 and their join touches g_3 at (0, 1, 0).

‡ Salmon-Fiedler: "Alg. der Lin. Trans.," p. 240.

This says that the points at which U meets J^2 are the points at which it meets H counted three times. But U touches H; therefore U meets J at three coincident points. Hence U is either a stationary tangent, a cusp tangent, or a nodal tangent of J. We can easily show that in some special case it is a nodal tangent; therefore it is so in general. This furnishes another proof of the proposition that the node of the cubic osculant of a tricuspidal quartic is at the point of contact.

The line equation of the conic point osculant, which we shall call H, at the point τ of Q is obtained by forming the Hessian of

$$a_1 \xi_1 \tau^4 + (b_1 \xi_1 + b_3 \xi_3) \tau^3 + c_3 \xi_3 \tau^2 + (d_2 \xi_2 + d_3 \xi_3) \tau + e_2 \xi_2 = 0.$$

The discriminant of the Hessian of a quartic is the product of the discriminant of the quartic and the square of the invariant g_3. *Therefore* H *touches* g_3 *as well as* Q. *Dr. Morley** *has shown that* H *touches a certain curve three times, which curve must be* g_3. H, *being also the conic osculant of the cubic osculant at the point* τ, *touches the stationary tangents of the cubic osculant which are lines of* g_3. *Hence* H *touches* g_3 *at the points where these stationary tangents touch it.*†

We have seen that the cubic osculant of Q at the point τ is a rational point cubic, touching Q at this point, touching the tangents to g_2 from this point and touching the six stationary tangents of Q. These eleven conditions determine the osculant uniquely, although they are not linear.

Let $U = 0$ be the point τ; *i. e.*,

$$U \equiv a_1 \xi_1 \tau^4 + (b_1 \xi_1 + b_3 \xi_3) \tau^3 + c_3 \xi_3 \tau^2 + (d_2 \xi_2 + d_3 \xi_3) \tau + e_2 \xi_2 = 0.$$

The sixteen common lines of the range of quartics $U g_3 + k H g_2 = 0$ *are the tangent to* H *at* U *counted twice, the two tangents from* U *to* g_2, *the six common tangents of* g_3 *and* H, *and the six common tangents of* g_3 *and* g_2. *The six common tangents of* g_3 *and* H *are the three stationary tangents of the cubic osculant at* U, *each counted twice. These lines, since they are stationary tangents of the osculant, will each count twice as a common tangent of the osculant and any other curve. Hence the cubic osculant belongs to this range when written in line form.*

Taking a curve in the form $x_i = (\alpha_i t)^n$, the first osculant is

$$x_i = (\alpha_i \tau)(\alpha_i t)^{n-1},$$

which meets the line $(x \xi) = 0$ when

$$((\alpha_i \tau)(\alpha_i t)^{n-1} \xi_i) = 0.$$

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* *Trans. Am. Math. Soc.*, Vol. VIII, p. 19.

† Each line of g_3 is a stationary tangent of two cubic osculants. Hence two conic osculants touch g_3 at each point and g_3^2 appears as part of the envelope of H.

Obviously the process of finding the line equation of the osculant is the same as that of finding the Steinerian of $((a_i t)^n \xi_i) = 0$.*

The highest power of τ involved in the line equation of the cubic point osculant of Q obtained in the manner just indicated is τ^4. If we eliminate t from the parametric line equations of the osculant (page 217), τ will occur as high as the sixteenth power in the resultant.

However, σ_τ (and consequently the equation of the twelfth degree in τ giving the parameters of the points of contact of the twelve cuspidal osculants) is obviously a factor of the equation so obtained.

These twelve osculants are curves of the third class.

The sextic covariant of $U = 0$ does not give proper osculants of Q. This covariant gives, neglecting the factor 54,

$$\begin{aligned}
J \equiv \, & (8\, a_1^2 \xi_1^2 (d_2 \xi_2 + d_3 \xi_3) - 4\, a_1 c_3 \xi_1 \xi_3 (b_1 \xi_1 + b_3 \xi_3) + (b_1 \xi_1 + b_3 \xi_3)^3)\, \tau^6 \\
& + 2\, (16\, a_1^2 e_2 \xi_1^2 \xi_2 + 2\, a_1 \xi_1 (b_1 \xi_1 + b_3 \xi_3)(d_2 \xi_2 + d_3 \xi_3) - 4\, a_1 c_3^2 \xi_1 \xi_3^2 \\
& \qquad\qquad + c_3 (b_1 \xi_1 + b_3 \xi_3)^2 \xi_3)\, \tau^5 \\
& + 5\, (8\, a_1 e_2 \xi_1 \xi_2 (b_1 \xi_1 + b_3 \xi_3) - 4\, a_1 c_3 \xi_1 \xi_3 (d_2 \xi_2 + d_3 \xi_3) \\
& \qquad\qquad + (b_1 \xi_1 + b_3 \xi_3)^2 (d_2 \xi_2 + d_3 \xi_3))\, \tau^4 \\
& - 20\, (a_1 \xi_1 (d_2 \xi_2 + d_3 \xi_3)^3 - e_2 \xi_2 (b_1 \xi_1 + b_3 \xi_3)^3)\, \tau^3 \\
& - 5\, (8\, a_1 e_2 \xi_1 \xi_2 (d_2 \xi_2 + d_3 \xi_3) - 4\, c_3 e_2 \xi_2 \xi_3 (b_1 \xi_1 + b_3 \xi_3) \\
& \qquad\qquad + (b_1 \xi_1 + b_3 \xi_3)(d_2 \xi_2 + d_3 \xi_3)^2)\, \tau^2 \\
& - 2\, (16\, a_1 e_2^2 \xi_1 \xi_2^2 + 2\, e_2 \xi_2 (b_1 \xi_1 + b_3 \xi_3)(d_2 \xi_2 + d_3 \xi_3) - 4\, c_3^2 e_2 \xi_2 \xi_3^2 \\
& \qquad\qquad + c_3 (d_2 \xi_2 + d_3 \xi_3)^2 \xi_3)\, \tau \\
& - (8\, e_2^2 \xi_2^2 (b_1 \xi_1 + b_3 \xi_3) - 4\, c_3 e_2 \xi_2 \xi_3 (d_2 \xi_2 + d_3 \xi_3) + (d_2 \xi_2 + d_3 \xi_3)^3) = 0.
\end{aligned}$$

For the quartic $t^2 (a_0 t^2 + 4\, a_1 t + 6\, a_2) = 0$ we find

$$J \equiv -(3\, a_0 a_2 - 2\, a_1^2)(a_1 t + 3\, a_2)\, t^5 = 0.$$

Hence J vanishes identically if U is the product of two squares. If α counts twice as a root of U, α is to be counted five times as a root of J, the sixth root of J being the fourth harmonic of α as to the two other roots of U.

Hence each of the seven† *curves obtained by equating to zero the coefficients of* τ^6, τ^5, *etc., in* J, *touches the four bitangents of* Q*; also* J *touches* Q *at* U.

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* I take pleasure in thanking Dr. J. R. Conner for noting that the curve of the above range which is the osculant must coincide with the Steinerian of $((a_i t)^n \xi_i) = 0$, and the obvious extension to curves of higher order.

† These seven curves can not be independent; and calculation shows that J is apolar to the sextic giving the parameters of the points of inflexion of Q for all values of ξ_i.

We know that there exists a syzygy,

$$J^2 \equiv g_2 H U^2 - g_3 U^3 - 4H^3.^*$$

Hence the six tangents from U to J^2 are the tangents from U to H counted three times. But U is on H, therefore the three tangents from U to J coincide. This means that the tangent to J at U is either a cusp tangent, a stationary tangent or a bitangent. It is easy to show that in some special case it is a bitangent; hence it is so in general.

J and Q have eighteen common tangents. If we substitute in J the values for ξ_i given in the line equations of Q (page 217), we have an equation of the eighteenth order in t and the sixth order in τ.

The octavic in t giving the points of contact of the bitangents of Q is a factor of this, the remaining factor being of the tenth order in t and the sixth order in τ. Of this, $(t-\tau)^5$ must be a factor, which says that the bitangent of J counts five times as a common tangent. Since this line is an ordinary line of Q, J and Q have four coincident common tangents at U.†

The remaining factor is of the fifth order in t and linear in τ. We know that if we ask that the point τ of Q shall be on the tangent to Q at the point t, the equation will contain $(t-\tau)^2$ as a factor. The remaining factor will be of the form

$$(\beta_0 t)^4 \tau^2 + 2(\beta_1 t)^4 \tau + (\beta_2 t)^4 = 0.$$

The fourth harmonic of t as to the roots of this equation in τ will be of the form

$$(\gamma_0 t)^5 \tau + (\gamma_1 t)^5 = 0.$$

Evidently this is the equation which gives the five other common tangents of J and Q.

§ 6. *The Meaning of the Disappearance of Terms.*

If k_i and m_i be respectively the coefficients of t^k and t^m in $(\alpha_i t)^n$, it is evident that if the relation $k_i = \lambda m_i$ exists for one triangle of reference, it will hold for all triangles of reference. If, then, by a change of parameter, we can bring the equations of a curve to a form in which such a relation exists, either the form is one to which the equations of all rational curves of the given order may be reduced, or the relation involves some projective property of the curve.

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[*] Salmon-Fiedler: "Alg. der Lin. Trans.," p. 264.

[†] Stated dually, Q passes through a node of J, meeting one branch of J in four consecutive points.

Thus the curve

$$x_i = a_i(t^n + \lambda t^{n-1}) + c_i t^{n-2} + \dots + p_i$$

may, by a change of parameter so that $t = \tau - \frac{\lambda}{n}$, be put in the form

$$x_i = a_i \tau^n + c_i' \tau^{n-2} + d_i' \tau^{n-3} + \dots + p_i',$$

which, if we change the reference triangle so that

$$y_1 = |\, x\, a\, c' \,|, \quad y_2 = |\, x\, a\, d' \,|, \quad y_3 = x_3,$$

becomes

$$\begin{aligned} y_1 &= d_1'' \tau^{n-3} + \dots, \\ y_2 &= c_2'' \tau^{n-2} + e_2'' \tau^{n-4} + \dots, \\ y_3 &= a_3 \tau^n + c_3' \tau^{n-2} + d_3' \tau^{n-3} + \dots, \end{aligned}$$

whence we see that the curve has a cusp at $\tau = \infty$.

Obviously, if a similar relation had existed between the coefficients of t and the constant terms in $(\alpha_i t)^n$, the curve would have had a cusp at $t = 0$, since, if we write $t = \frac{1}{t}$, we merely interchange the coefficients of t^k and t^{n-k}. Conversely, if a curve has a cusp we may write its equations

$$\begin{aligned} x_1 &= d_1 t^{n-3} + \dots, \\ x_2 &= c_2 t^{n-2} + \dots, \\ x_3 &= a_3 t^n + b_3 t^{n-1} + c_3 t^{n-2} + \dots, \end{aligned}$$

which, by a change of parameter leaving ∞ unaltered, may be put in a form in which $b_3 = 0$.

In this form one set of the I_n is $t = 0$. Hence it follows that, if a curve has a cusp at $t = \alpha$, one set of the I_n is $(t - \alpha)^{n-1}(t - \beta) = 0$. Hence, with each cusp α of a plane rational curve there is associated a definite point, β, of the curve.*

On the other hand, the equations of the general rational cubic may be put in the form

$$x_1 = t^3, \quad x_2 = 1, \quad x_3 = t^2 + \lambda t,$$

and for all reference triangles $c_i = \lambda b_i$. In this case the relation has no projective meaning.

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* Dually with each stationary tangent, a, there is associated a definite line, β. Thus, for Q we see by inspection that $c_3 t^6 + 3 d_3 t^5 = 0$ is apolar to the expressions for ξ_i in the line equations of the curve (page 217). If, in the equations of the point cubic osculant at the point $t = 0$, we let $t = -\frac{3 d_3}{c_3}$, we have the coordinates of the tangent to the osculant from the point of contact.

On the Primitive Groups of Classes Six and Eight.[*]

By W. A. Manning.

In the *Comptes Rendus* of Dec. 23, 1872, page 1754, Jordan published the results of his investigation of the primitive groups of the first 13 classes. The calculations by which these results were reached were "long and laborious" and have not been published. In view of the fundamental importance of the primitive groups, it seems desirable that these proofs should be supplied. This was done in part by Jordan[†] when he determined all the primitive groups of class p (p a prime number), and by the present writer[‡] in the determination of the primitive groups of class $2p$, which contain a substitution of order p and degree $2p$. Netto[§] and Miller[||] have considered the groups of class 4, and their results agree with Jordan's. The primitive groups of class 6, which do not contain a substitution of order 3 on 6 letters, and the primitive groups of class 8 are the object of investigation in this paper. It is found that there are in all 14 groups of class 6 and 18 groups of class 8. Jordan's lists give 13 groups of class 6 and 15 of class 8.

The following theorem is restated in altered form from a memoir by Jordan.[¶]

Theorem I. Let A, A', ... be a complete set of conjugate substitutions of prime order p and degree qp in a primitive group G. The set A, ... generates a transitive group H. Let I be an intransitive subgroup of H generated by a certain number of the substitutions A, ... Let there be a set of intransitivity (a, ...) in I such that none of the substitutions A, ... transform all the letters

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[*] Read before the American Mathematical Society (San Francisco Section), Dec. 19, 1903.

[†] C. Jordan: *Liouville's Journal*, Ser. 2, Vol. XVII (1872), p. 363.

[‡] *Transactions of the American Mathematical Society*, Vol. IV (1903), p. 351. See also "On the Groups of Class Ten," American Journal of Mathematics, Vol. XXVIII (1906), p. 226.

[§] Netto: "Theory of Substitutions," Cole's translation (1892), p. 133.

[||] Miller: *American Mathematical Monthly*, Vol. IX (1902), p. 63.

[¶] *Crelle's Journal*, Vol. LXXIX (1874), pp. 249–253.

a of that set into other letters. Let $b, \ldots.$ be the letters of the other sets of intransitivity of I and let $c, \ldots.$ be the letters of H not displaced by I. Then in H there is always a substitution B similar to A which connects a and b transitively, and has not more than one letter c in each of its q cycles. This happens notably when the number of letters in the set a exceeds $q\,\frac{p-1}{2}$, if p is odd, or q, if $p = 2$.

COROLLARY I. If $p = 2$, and the number of letters $a, \ldots.$ is exactly q, the theorem clearly holds unless, perhaps, when all the substitutions which replace the letters $a, \ldots.$ by other letters are of the form $C = (a_1c_1)(a_2c_2)\ldots.(a_qc_q)$. When, in this exceptional case, G is of class $2q$, any substitution A_1, similar to A, of I is of the form $(a_1a_2)\ldots.(b_1b_2)\ldots.$ where at least half the letters are $b, \ldots.$. But $A_1CA_1C = (a_1a_2)\ldots.(c_1c_2)\ldots.$ is of degree not greater than $2q$. Hence we conclude that every substitution similar to A in I, which displaces any of the letters $a, \ldots.$, displaces all of the q letters $a, \ldots.$. This exceptional case can not arise in a group of class $2q$ unless q is even. We also note that C and A_1CA_1C displace exactly the same $2q$ letters.

COROLLARY II. Suppose that there are just $q - 1$ letters $a\ldots.$. We can always find a substitution C similar to A connecting $a\ldots.$ and $b\ldots.$ and having in no cycle more than one new letter c unless *any* substitution of H, similar to A, which replaces all the letters $a\ldots.$ by other letters, has one of the four following forms:

$$
\begin{aligned}
C_1 &= (a_1c_1)\ldots.(a_{q-1}c_{q-1})(c_qc_{q+1}),\\
C_2 &= (a_1c_1)\ldots.(a_{q-1}c_{q-1})(b_1c_q),\\
C_3 &= (a_1c_1)\ldots.(a_{q-1}c_{q-1})(b_1b_2),\\
C_4 &= (a_1b_1)\ldots.(a_\lambda b_\lambda)(a_{\lambda+1}c_1)\ldots.(a_{q-1}c_{q-1-\lambda})(c_{q-\lambda}c_{q-\lambda+1}),
\end{aligned}
$$

where

$$1 \overline{<} \lambda \overline{\overline{<}} q - 1.$$

It will be shown that C_1 and C_2 never occur in H, and that C_3 occurs only when *any* substitution of I, which is similar to A, contains both b_1 and b_2 in different cycles, and displaces all the $q - 1$ letters $a\ldots.$ when q is odd, or all but one when q is even.

Let $A_1 = (a_1a_2)\ldots.(b_1b_2)\ldots.$ be a substitution of I similar to A. The number of letters $b\ldots.$ in A_1 is at least $q + 1$. Then $A_1C_1A_1C_1$ is of degree less than $2q$ and is not the identity. If A_1 does not displace the letter b_1 of C_2, the same is true of $A_1C_2A_1C_2$. Let A_1 displace b_1. Then $A_1C_2A_1C_2$ is at most of degree

$2q + 1$ and is non-regular, having one cycle of order 3 and the remaining cycles of order 2. But in a group of class u, the substitutions of degree u and $u + 1$ are regular. Passing to C_3 it is clear that A_1 must contain both b_1 and b_2, and in different cycles. We write $A_1 = (a_1a_2)\dots(b_1b')(b_2b'')$ and have

$$C_3A_1C_3 = (c_1c_2)\dots(b_2b')(b_1b'')$$

so that

$$D = A_1C_3A_1C_3 = (a_1a_2)\dots(c_1c_2)\dots(b_1b_2)(b'b'').$$

If q is odd D is of degree $2q + 2$, and if q is even, D is of degree $2q$.

It is easy to show, in the same way, that in C_4 λ is greater than 2, except possibly in the case of $\lambda = 2$ and $q = 5$, when there may be a substitution of degree 10 and order 5.

Class 6.

Since the primitive groups of class 6 which contain a positive substitution on 6 letters have been determined,* when the groups of class 6 are referred to in what follows we shall have in mind only those which include no positive substitution on 6 letters.

The group G contains by hypothesis a substitution A of the form

$$a_1a_2 \,.\, b_1b_2 \,.\, c_1c_2 .$$

There is a self-conjugate, and hence transitive, subgroup H generated by all the substitutions similar to A. The degree of H is the same as that of G and hence H must displace more than 6 letters. Then some substitution of the series of conjugates $A, \dots$ must displace one or more letters new to A. Again one of these generators must connect transitively the letters of A and new letters. If A' is such a substitution, it can not displace more than three letters new to A. Hence G always contains a diedral rotation group of degree 7, 8, or 9. We shall consider these three cases in succession and first merely determine from each of them certain transitive subgroups of H.

The diedral rotation group of degree 7 is itself a primitive group of order 14.

In the second case the group $\{A, A'\}$ is of degree 8 and the substitution AA' is of degree 8 also. This product may be of order 4 or 2.

Let the product AA' be of order 4. The group $J = \{A, A'\}$ is an intransitive octic group. By Theorem I we can find a substitution A'' connecting the two sets of intransitivity of J and introducing at most two new letters. Now G

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* MANNING: *loc. cit.*

can not contain two substitutions of class 6 on exactly the same letters, for the product of two such would be a positive substitution of degree 6. Then A'' displaces at least one letter new to J.

If A'' displaces just one new letter, *the group* $K = \{A, A', A''\}$ *is a simply transitive primitive group of degree* 9. The positive subgroup is of class 8 and order 36, so that K has an invariant regular non-cyclic subgroup of order 9.

Again, suppose A'' to introduce two new letters; K is a transitive group of degree 10. Clearly K can not be primitive. Nor can K have two systems of 5 letters each. If there are 5 systems of two letters each, K is isomorphic to the symmetric group on 5 letters and to cycles of three letters in it must correspond positive substitutions of degree 6. This is inadmissible.

The product AA' may be of order 2. Then A and A' are commutative so that A' may be written $a_1b_1 \, . \, a_2b_2 \, . \, \alpha_1\alpha_2$. There will be another substitution A'' connecting the set $a_1, \ldots$ with another set, and bringing in one or two new letters. Let A'' bring in just one new letter β. Then we can write at once $A'' = a_1c_1 \, . \, a_2c_2 \, . \, \alpha_1\beta$; for if two substitutions of class 6 have just three letters in common their product must be of order 3. This means that A'' and A have just four letters in common and their product is of order 2. The group $\{A, A', A''\}$ is of order 12 and has two sets of intransitivity. The substitution A''' can be written either $a_1\alpha_1 \ldots$ or $a_1\alpha_2 \ldots$. By the principle just employed it is clear that we get either $A''' = a_1\alpha_1 \, . \, b_1\alpha_2 \, . \, c_1\beta$ or $A''' = a_1\alpha_2 \, . \, b_1\alpha_1 \, . \, c_2\gamma$. In the first case we obtain an imprimitive group of order 36 which is a direct product of two symmetric groups, and in the second case a primitive group of order 120 isomorphic to the symmetric group on 5 letters.

Going back to the group $\{A, A'\}$ we see that A'' can not have two new letters in different cycles. For we may write $A'' = a_1c_1 \ldots$, and since the product AA'' can not be of order 3 it is of degree 8 and order 4 or 2, which is impossible.

It remains to consider the diedral group of degree 9 generated by A and a substitution of the form $a_1a_3 \, . \, b_1b_3 \, . \, c_1c_3$. Here there are three sets of intransitivity of three letters each. By Corollary I to Theorem I there is a substitution A'' in the series $A, \ldots$ of the form $a_1b_1 \ldots$ or $a_1b_2 \ldots$. It is not necessary to consider $a_1b_3 \ldots$, since it is the transform of $a_1b_2 \ldots$ by

$$a_2a_3 \, . \, b_2b_3 \, . \, c_2c_3.$$

But $\{A, A''\}$ can only be one of the fundamental diedral groups already discussed.

Summing up, we have 4 fundamental transitive groups, one of which must be a subgroup of any primitive group of class 6 not containing positive substitutions on 6 letters. We shall now take up these subgroups in detail.

1. *The* G^7_{14}.

Since all the transitive groups of degree 7 and class 6 are included in the metacyclic group, this G^7_{14} is not included in a larger primitive group of the same degree. We ask if G^7_{14} can be the maximal subgroup of a $G^8_{8.7.2}$. The negative substitutions of degree 8 in G^8 must be of order 8. Then among the positive substitutions of G^8 are some of order 4. But the positive subgroup of G^8 is of class 7 and has no operators of order 4.

2. *The* G^9_{72}.

The 12 substitutions of class 6 in G^9_{72} are made up of the 36 possible combinations of 9 letters, two at a time, so that a G of degree 9 including G^9_{72} can have no other substitution of class 6. Then G^9_{72}, if contained in a larger group G' of degree 9, is invariant. Let G'_1 and G_1 be the subgroups leaving one letter fixed of G' and G^9_{72} respectively. The subgroup G'_1 contains G_1 invariantly, and has negative substitutions other than the 4 of degree 6. They can only be cycles on 8 letters. Then G'_1 is transitive and of order 16. If the positive subgroup of G'_1 is not the quaternion group, G' contains more than 9 substitutions of degree 8 and order 2, and since from 9 letters only 36 transpositions can be formed, two of these substitutions would have a transposition in common, so that their product is a positive substitution of degree less than 8, an impossibility. Now the quaternion group is completely determined by a subgroup of order 4, so that G'_{144}, if it exists, is unique. It does exist and is a well-known group. It may be generated by 12 . 34 . 68, 13 . 56 . 78, 15 . 47 . 89, and 1537 . 2846. We now show that G^9_{72} can not be the subgroup leaving one letter fixed in more than one G^{10}_{720}. In G^{10}_{720} there will be a substitution $B = (1\alpha)(\,.\,.)(\,.\,.)$ similar to A. Unless B is commutative with the 4 substitutions 24 . 58 . 67, 25 . 37 . 69, 28 . 39 . 45, and 36 . 48 . 79, the commutator of B and one of them will be of degree 7 or less, and positive. Then $B = 1\alpha\,.\,25\,.\,48$ or $1\alpha\,.\,39\,.\,67$; but $24\,.\,58\,.\,67 \times 28\,.\,39\,.\,45 \times 1\alpha\,.\,39\,.\,67 = 1\alpha\,.\,39\,.\,67$, so that G^{10}_{720} is unique. The holomorph of the modular group of degree 10 and order 360 is of order 1440 and is triply transitive on 10 letters. The subgroup leaving three letters fixed is of order 2 and degree 6. The quotient group with respect to the modular group is the axial group. Corresponding to the three subgroups of order 2 in the quotient group are three subgroups of order 720, one positive of class 8, one positive and negative of class 8, and one positive and negative of class 6. This last group contains G^9_{72}. Since G^9_{72} completely determines G^{10}_{720},* G^{10}_{1440} is completely determined by G^9_{144}. Since neither the modular nor the Mathieu group of degree p^n+1 is maximal in a group of degree p^n+2, this chain of primitive groups ends here.

3. *The Imprimitive H^9_{36}.*

This group is a direct product of two non-Abelian groups of order 6. Its systems of imprimitivity of three letters each can be chosen in only two ways. Any larger group of degree 9 containing H^9_{36} must transform it into itself, since another substitution of class 6 which does not lead back to a previously considered case can not be found. Let G be this larger group and G_1' its subgroup leaving one letter fixed. In G_1 there must be a cycle on 8 letters, which will permute the two substitutions of class 6. Then the square of this cycle transforms a substitution of class 6 into itself. This is an absurdity. Since the systems of imprimitivity of H can not be chosen in 4 ways,† there only remains the possibility that H is the maximal subgroup in a doubly transitive group G on 10 letters of order 360. But this is impossible, for the positive subgroup of order 180 would have 36 subgroups of order 5, and 45 subgroups of order 2, making at once 189 operators.

4. *The G^{10}_{120}.*

If this primitive G_{120} is contained in a larger group G' of the same degree it is invariant in it, since another substitution of class 6 on these letters can not exist. The subgroup G_1', leaving one letter of G' fixed, transforms G_1, the maximal subgroup of G^{10}_{120}, into itself, and is intransitive in consequence. Since the constituent of degree 3 is symmetric, an operator of G_1' not in G_1 would lead to another substitution of class 6, which is impossible. Another substitution B, similar to A, introducing one new letter δ, can not be determined subject to the conditions under which $A''' = a_1\alpha_2 . b_1\alpha_1 . c_2\gamma$ was determined. Hence we can not pass to a group of degree 11 from G^{10}_{120}.

There are in all 14 primitive groups of class 6, of which 8 contain a substitution of order 3 on 6 letters, and 6 do not. The latter are the G^7_{14}; G^9_{72}, G^9_{144}, G^{10}_{720}, G^{10}_{1440}, and G^{10}_{120}.

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* De Seguier: *Comptes Rendus*, Vol. CXXXVII (1903), p. 37.

† *Transactions of the American Mathematical Society*, Vol. VII (1906), pp. 499-508.

Class 8.

Every group of class 8 contains a substitution $A = a_1a_2 . b_1b_2 . c_1c_2 . d_1d_2$. Any primitive group of class 8 contains a transitive subgroup generated by substitutions similar to A. We shall first determine these subgroups and then take up each one of them in detail, seeking to find in what primitive groups, if any, they may be contained.

If A_1 is a substitution similar to A introducing no new letter, the product AA_1 is of order 4 or 2.

Let $(AA_1)^4 = 1$; then $\{A, A_1\}$ *is a regular octic group.*

Let $(AA_1)^2 = 1$, so that $\{A, A_1\}$ is an axial group. There may or may not be a substitution A_2 similar to A on these same letters. Suppose that A_2 exists and that it is commutative with both A and A_1. Then $\{A, A_1, A_2\}$ *is the regular Abelian group of order* 8, *all of whose substitutions are of order* 2. If A_2 is not commutative with both A and A_1, it must with one of them generate the regular octic group, given above.

We now assume that besides the three substitutions

$$a_1a_2 . b_1b_2 . c_1c_2 . d_1d_2, \quad a_1b_1 . a_2b_2 . c_1d_1 . c_2d_2, \quad a_1b_2 . a_2b_1 . c_1d_2 . c_2d_1,$$

of the axial group, the series of similar substitutions $A, \ldots.$ includes none displacing these 8 letters only. By the Corollary I to Theorem I, the series $A, \ldots.$ contains a substitution A_2 connecting transitively the two sets of letters in $\{A, A_1\}$ and introducing at most three new letters, with not more than one new letter in any cycle, unless there occurs in $A, \ldots.$ a substitution B of the form $a_1\alpha_1 . a_2\alpha_2 . b_1\alpha_3 . b_2\alpha_4$. This exception need not here be considered, since the group $\{B, (BA)^2, (BA_1)^2\}$ is the regular group all of whose operators are of order 2. If A_2 introduces one new letter, $\{A, A_1, A_2\}$ is a transitive group of degree 9, which contains invariantly a regular group of order 9. This is impossible, since the group of isomorphisms of neither of the two groups of order 9 contains an axial subgroup of class 8. Let A_2 introduce two new letters. Then $J = \{A, A_1, A_2\}$ is transitive. It can contain no substitutions of order 3, and hence is of order 40 or 80. Then it has an invariant subgroup of order 5 and degree 10. But the largest group on 10 letters which contains no cycles of 5 letters and which transforms this subgroup into itself is of order 40 and has negative substitutions. If A_2 introduces three new letters, one of the three products $A_2A, \ldots.$ contains a cycle of 4 letters and is of degree 11, an impossibility in a group of class 8.

In what follows it is assumed that the set $A, \ldots$ contains no two substitutions on exactly the same letters.

For convenience of reference we shall now state some principles of which continual use will be made.

No substitution similar to A can have just one letter in common with A; nor may it have just two letters in common with A unless it is commutative with A. Suppose that B, similar to A, has just three letters in common with A.[*] The common letters must belong to different cycles in both A and B in order that $(AB)^2$ may not be of class less than 8. But when this condition is satisfied, B is of the form $\alpha_1\alpha_2 . \alpha_3a_1 . \alpha_4b_1 . \alpha_5c_1$ and $(AB)^3$ is of class 4.

If A and B have just 4 letters in common, they generate a positive group, and the product AB can only be of order 4, 3, or 2. The positive intransitive diedral groups in which AB is of degree 12 and orders 10, 8, and 6 are not generated by two substitutions of degree 8. We reject $\{A, B\}$, where $(AB)^4 = 1$, when it is a (4, 1) isomorphism between the intransitive $DG_8^{8,4}$,[†] itself a simple isomorphism, and a G_2^4, because a $DG_4^{8,8}$ is included. With the product AB still of order 4 and degree 12, $\{A, B\}$ may be a (2, 1) isomorphism between the intransitive $DG_8^{8,6}$ and the intransitive $DG_4^{4,2}$. We may write B, for example, as $a_1b_1 . c_1\alpha_1 . c_2\alpha_2 . \beta_1\beta_2$, and hence $(AB)^2 = a_1b_1 . a_2b_2 . c_1c_2 . \alpha_1\alpha_2$. If AB is of order 3, $\{A, B\}$ is a simple isomorphism between four groups $DG_6^{8,2}$. If B is commutative with A, two cases are to be distinguished as AB is of degree 8 or 12.

If A and B have 5 letters in common, AB is of order 6 and degree 11, or of order 3 and degree 9. When AB is of degree 11, the subgroup $\{A, BAB\}$ is a diedral group of degree 9, class 8, and order 6.

Should the two similar substitutions A and B have just 6 letters in common, the product AB may be of order 5, 4, or 2. The product can not be of order 8, for then $\{A, B\}$ would include the regular octic.

Among the conjugates of A there must exist at least one substitution A_1 which connects old and new letters. The degree of $\{A, A_1\}$ is equal to or less than 12.

Let A_1 introduce just one new letter. If AA_1 is of order 9, $\{A, A_1\}$ *is a transitive diedral rotation group of order* 18.

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[*] BOCHERT: *Mathematische Annalen*, Vol. XL (1892), p. 176.

[†] By this notation is indicated a diedral rotation group of degree 8, class 4, and order 8.

The only remaining possibility is that AA_1 is of order 3, in which case $\{A, A_1\}$ is a simple isomorphism between the regular and non-regular representations of the non-Abelian group of order 6. By Theorem I the set $A, \ldots$ contains a substitution A_2 which connects these two transitive constituents.

Suppose first that A_2 displaces no letter not in $\{A, A_1\}$. Then in the transitive group $\{A, A_1, A_2\}$ any two substitutions of degree 8 and order 2 have just 7 letters in common and their product is of order 3. *Hence* $\{A, A_1, A_2\}$ *is of order* 18 *and contains an invariant non-cyclic group of order* 9.

Let A_2 introduce one new letter. Then $\{A, A_1, A_2\}$ is a primitive group of degree 10. The subgroup leaving one letter fixed must coincide with $\{A, A_1\}$, since it can not contain a subgroup of order 9 without including one of the two transitive groups of order 18 just determined. Hence $G = \{A, A_1, A_2\}$ is of order 60; whence it follows that G contains 6 conjugate subgroups of order 5, by means of which it may be represented as a doubly transitive group of degree 6 and class 4. *This is the icosaedral group which has one and only one primitive representation on* 10 *letters.* We note that the icosaedral group can not be written as an imprimitive group on 10 letters.*

Let $G = \{A, A_1, A_2\}$ be a transitive group of degree 11, with A_2 bringing in two new letters. As before, the order of G is not divisible by 9, so that it is at most $11 . 10 . 6 = 660$. That the order is exactly 660, is evident if we note that there must be 12 conjugate subgroups of order 11, each transformed into itself by 55 substitutions. This group, then, if it exists, is simply isomorphic to the doubly transitive group of class 10 on 12 letters, the modular group. Hence the subgroup of order 60 in G is icosaedral, and we are led back to the previous case.

If A_2 introduces 3 new letters, we may assume that it is of the form

$$a'd_3 . \alpha_1 — . \alpha_2 — . \alpha_3 —,$$

where d_3 is the new letter introduced by A_1, and a' is some one of the larger set of intransitivity of $\{A, A_1\}$; α_1, α_2, and α_3 are the 3 new letters. Now A_2 and A have just 4 letters in common and the group $\{A, A_2\}$ is of order 6 with 4 sets of intransitivity. Then A_2 displaces just one letter from each cycle of A. Then A_2 has just 5 letters in common with A_1 or else with A_1AA_1,—with A_1, say. But the product A_1A_2 can not be of order 3, since A_1 and A_2 have no common cycle, nor can it be of order 6 and degree 11, since $\{A_1, A_2\}$ can not have a transitive constituent of degree 2.

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* BURNSIDE: "Theory of Groups" (1897), p. 179.

We now return to A_1 and assume that it displaces two letters not in A. It has been remarked that the product AA_1 can be of order 5, 4, or 2. In the last case only (A and A_1 are commutative) do the two new letters α_1 and α_2 form a cycle of A_1. The group $\{A, A_1\}$* is

$$\begin{array}{ll} 1, & A_1 = a_1b_1 \,.\, a_2b_2 \,.\, c_1c_2 \,.\, \alpha_1\alpha_2, \\ A = a_1a_2 \,.\, b_1b_2 \,.\, c_1c_2 \,.\, d_1d_2, & A_1' = a_1b_2 \,.\, a_2b_1 \,.\, d_1d_2 \,.\, \alpha_1\alpha_2. \end{array}$$

Let A_2 be a substitution similar to A connecting the set of intransitivity of 4 letters in $\{A, A_1\}$ with one of the other sets. It may be written $a_1c_1 \ldots.$ without loss of generality.

Suppose first that A_2 displaces no letter new to $\{A, A_1\}$. In order that A_2 may not with A, A_1, or A_1' generate one of the diedral subgroups already considered, it must displace α_1, α_2, d_1, d_2, c_1, and c_2. Then A_2 leaves fixed either both b_1, b_2 or both a_2, b_2. It can not leave a_2 and b_1 fixed, for it is not commutative with A_1', as it would then have to be. Then put $A_2 = a_1c_1 \,.\, a_2c_2 \,.\, d_1d_2 \,.\, \alpha_1\alpha_2$; but $A_1'A_2 = a_1b_2c_1 \,.\, a_2b_1c_2$. If A_2 leaves a_2 and b_2 fixed, $A_2 = a_1c_1 \,.\, b_1c_2 \,.\, \alpha_1\alpha_2 \,.\, d_1d_2$ and $A_1'A_2 = a_1b_2c_1 \,.\, a_2c_2b_1$.

Let A_2 bring in one new letter. It may be assumed that $\{A, A_1, A_2\}$ is not transitive. The letters may fall in sets of intransitivity in the following 6 ways; (6, 3, 2), (6, 5), (7, 2, 2), (7, 4), (8, 3), (9, 2). The third and fourth cases may be immediately excluded since they include operators of order 7. In the first case the transitive constituents of degrees 6 and 3 must be in simple isomorphism: hence the constituent on 6 letters is regular and the entire group can not be of class 8. The second case must be a simple isomorphism between two representations of the icosaedral group or else include it. The class is 6. In the fifth case all the substitutions of class 8 will be in an invariant subgroup. In the last case there is an invariant subgroup in the first constituent corresponding to identity of the second. If this subgroup is of class 8, it includes one of the fundamental subgroups already discussed. If it is of class 9, the entire group is of order 6, which is here impossible.

Let A_2 contain two letters β_1 and β_2, not displaced by $\{A, A_1\}$. We may write $A_2 = a_1\alpha_1 \ldots.$ Suppose first that A_2 leaves α_2 fixed. Then $\{A, A_2\}$ is of degree 11 and order 6, and A_2 has a cycle in common with A. Comparing A_2 with A_1 and A_1', this is seen to be impossible. If A_2 displaces α_2, AA_2 is of order

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* This group need not be considered, as we might restrict A_1 to substitutions which connect old and new letters, but it is found convenient to use it.

The group $L = \{K, B_1\}$ is of order 120, and is a simple isomorphism between a $G^{5,2}_{120}$ and a $G^{10,6}_{120}$. The remaining 6 substitutions of class 8 are

$$a_1\gamma_1 \,.\, b_2\delta \,.\, c_1\alpha_1 \,.\, d_1\beta_2, \quad a_1\delta \,.\, b_2\gamma_1 \,.\, c_2\alpha_2 \,.\, d_2\beta_1,$$
$$a_2\gamma_1 \,.\, b_1\delta \,.\, c_2\alpha_1 \,.\, d_2\beta_2, \quad a_2\alpha_2 \,.\, c_1\delta \,.\, b_2\alpha_1 \,.\, d_2\gamma_2,$$
$$a_2\delta \,.\, b_1\gamma_1 \,.\, c_1\alpha_2 \,.\, d_1\beta_1, \quad \gamma_1\alpha_2 \,.\, b_1c_1 \,.\, b_2c_2 \,.\, \beta_2\gamma_2.$$

The subgroup of L leaving the letter a_1 fixed is intransitive in sets $d_1\beta_1$, $d_2\beta_2\gamma_2$, etc. Then it may be assumed that the substitution which connects the two sets of L is either

$$C_1 = a_1d_1 \,.\, a_2d_2 \,.\, \alpha_2\gamma_2 \,.\, \gamma_1\beta_2,$$

or

$$C_2 = a_1d_2 \,.\, a_2d_1 \,.\, \beta_1\delta \,.\, \alpha_1\varepsilon.$$

Now $\{L, C_1\}$ *is a primitive group of order* 720 *isomorphic to* (abcdef) all. In the group $\{L, C_2\}$ the conjugates of the product

$$\gamma_1\alpha_2 \,.\, b_1c_1 \,.\, b_2c_2 \,.\, \beta_2\gamma_2 \,.\, a_1d_2 \,.\, a_2d_1 \,.\, \beta_1\delta \,.\, \alpha_1\varepsilon$$

generate a self-conjugate regular Abelian subgroup of order 16, of type (1, 1, 1, 1). *This fundamental group can readily be identified with the primitive* $G^{16,8}_{1920}$ *given by* MILLER.*

We return to the group K. The group $\{K, B_2\} = L$ is of order 36 and has one set of intransitivity of degree 9, and two of degree 3. There is one other substitution of class 8: $a_2\alpha_1 \,.\, c_2\gamma_1 \,.\, b_2\alpha_2 \,.\, d_2\delta_1$. Consider now the two possible substitutions

$$C_1 = a_1\beta_1 \,.\, b_1\beta_2 \,.\, c_1\gamma_2 \,.\, d_1\varepsilon,$$

and

$$C_2 = a_1d_1 \,.\, a_2d_2 \,.\, \alpha_1\delta \,.\, \beta_1\varepsilon.$$

Since $s = \beta_1d_1 \,.\, \beta_2d_2 \,.\, a_2b_1 \,.\, c_1\alpha_1 \,.\, c_2\alpha_2 \,.\, \gamma_2\delta$ gives $sC_1s = C_2$ and $sLs = L$, only C_1 need be used. One constituent of $M = \{L, C_1\}$ is the symmetric group of degree 4 and the other is an imprimitive group on 12 letters, whose 4 systems are permuted according to the symmetric group. The order of M is 144. The new substitutions of class 8 are: $a_2\beta_1 \,.\, b_2\beta_2 \,.\, c_2\gamma_2 \,.\, d_2\varepsilon$ and $\alpha_1\beta_1 \,.\, \gamma_1\gamma_2 \,.\, \alpha_2\beta_2 \,.\, \delta_1\varepsilon$. There is only one form in which D_1 may be assumed:

$$D_1 = a_1d_1 \,.\, a_2d_2 \,.\, \alpha_1\delta \,.\, \beta_1\varepsilon.$$

The group $H = \{M, D_1\}$ *is imprimitive of degree* 16 *and order* 576. It is the direct product of two symmetric groups of order $\underline{|4}$.

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*MILLER: AMERICAN JOURNAL OF MATHEMATICS, Vol. XX (1898) p. 229.

It may now be assumed that the series $A, \ldots.$ contains no substitution which connects the set $a_1 \ldots.$ of K with $\alpha_1 \ldots.$ or $\beta_1 \ldots.$. Let us consider $\{K, B_3\}$. This group has two sets of intransitivity, so that a substitution of the form $a_1 x \ldots.$ exists, where x is a letter of the set $\alpha_1 \ldots., \beta_1 \ldots.,$—contrary to hypothesis. Finally $\{K, B_4\}$ leads uniquely to the substitution $a_1\delta_1 . d_1\alpha_1 . b_2\gamma_2 . c_2\beta_2$, which connects the sets $a_1 \ldots.$ and $\beta_1 \ldots.$.

In conclusion we have to consider $A_1 = a_1\alpha_1 . b_1\alpha_2 . c_1\alpha_3 . d_1\alpha_4$, and the group $\{A, A_1\}$. But by Corollary II to Theorem I, we know that there is in the series $A, \ldots.$ a substitution $A_2 = a_1 b' \ldots.,$ where b' is one of the 3 letters b_1, b_2, α_2. Hence this case is included in one of the preceding.

Nine distinct transitive groups have been determined, one of which is included in any primitive group of class 8. We shall take up these subgroups one by one and find what primitive groups are determined by them. In discussing a particular subgroup use may again be made of any special theorems or processes employed in obtaining it originally.

1. *The Regular Octic.*

Our primitive group G contains the regular octic (I). Since $A, \ldots.$ generate a transitive group of degree greater than 8, there is a substitution B similar to A which connects some new letters transitively with the letters of this octic group. Now $\{I, B\}$ can not be a transitive group of degree 9, for I is not a subgroup of the group of isomorphisms of a group of order 9. If I is contained in a primitive group of degree greater than 9, it is contained in an imprimitive group of degree 10 and order 80. That it can not be contained in an imprimitive group of degree 12, maximal in a primitive group, results from the fact that of the three subgroups of order 4 in I one is characteristic, so that there are but two interchangeable systems of 4 letters each with one letter in common. Now the group of order 80 can not have an invariant subgroup of order 5 since the largest group of degree 10 and class 8 in which a subgroup of order 5 is invariant is of order 40. Nor, with 35 substitutions displacing 8 letters, can there be 16 subgroups of order 5 and degree 10.

2. *The Regular Abelian G_8^8 of Type* (1, 1, 1.)

This case is based upon the regular group (I) all of whose substitutions are of order 2. It may be written:

$$\begin{array}{ll} 1, & A_1 = 12 \,.\, 34 \,.\, 56 \,.\, 78, \\ A_2 = 13 \,.\, 24 \,.\, 57 \,.\, 68, & A_3 = 14 \,.\, 23 \,.\, 58 \,.\, 67, \\ A_4 = 15 \,.\, 26 \,.\, 37 \,.\, 48, & A_5 = 16 \,.\, 25 \,.\, 38 \,.\, 47, \\ A_6 = 17 \,.\, 28 \,.\, 35 \,.\, 46, & A_7 = 18 \,.\, 27 \,.\, 36 \,.\, 45. \end{array}$$

Since the group of isomorphisms of the non-cyclic group of order 9 does not include I, the new substitution B, connecting new letters transitively with the letters of I, can not introduce just one new letter.

If B introduces just two new letters, $\{I, B\}$ is imprimitive of order 80. There are 35 substitutions of class 8 and 44 of degree 10, the latter in 11 subgroups of order 5, since $\{I, B\}$ is positive. But 11 is not a divisor of 80.

If B brings in three new letters, and has not already two new letters in a cycle, i can be chosen so that BA_iB has one cycle of new letters. Then $\{I, B\}$ includes a subgroup of degree 9 and class 8, an impossibility.

Let B ntroduce 4 new letters. The new letters must be in different cycles of B. The 4 old letters of B must unite by twos in three of the substitutions of I and must be in different cycles in the remaining 4. Since the holomorph of I is triply transitive, we can write $B = 1\alpha \,.\, 2\beta \,.\, 3\gamma \,.\, 4\delta$, the element 4 being fixed by the remark just made. The group $J = \{I, B\}$ if of order 96, is 3-fold imprimitive on 6 systems. It can not be contained in a primitive group G of degree 12. Nor can a substitution C similar to A be found introducing one new letter. We would have $C = 1x \,.\, \text{—}$ containing three of the letters $\alpha, \beta, \gamma, \delta$, and three of the letters 5, 6, 7, 8, as is seen by comparing with $B'IB'$, where $B' = BA_4B$. Then $CA_1C = x2 \,.\, 34 \,.\, 5'\alpha' \,.\, \beta'\gamma'$ and $(A_1C)^2$ is of degree 8 or 10 with the cycle $(1x2)$, which is impossible. It is evident that G can not contain a substitution or subgroup on 13 letters, so that if C introduces two new letters $\{I, C\}$ is imprimitive with 7 systems. Since the group according to which these systems are permuted can not be of class less than 4 and certainly is of class 4, the systems go among themselves according to the simple $G^{7,4}_{168}$. In the three ways of dividing the letters of J^{12}_{96} into systems α is associated with β, γ, δ respectively. The systems of $K = \{J, C\}$ must conform to one of these three different groupings. Let us suppose for the moment that α, β form a system of imprimitivity in K. Then the substitution C can be written $\alpha x \,.\, \beta y \ldots$, since K is doubly transitive in the systems. Further, since I is transitive, $C = \alpha x \,.\, \beta y \,.\, 1\text{—} \,.\, 2\text{—}$, if we choose. Comparing C with $1\alpha \,.\, 2\beta \,.\, 3\gamma \,.\, 4\delta$ it is seen that the two remaining elements belong to 5, 6, 7, 8. Then

$$C = \alpha x \,.\, \beta y \,.\left\{\begin{matrix} 15 \,.\, 26 \\ 16 \,.\, 25 \\ 17 \,.\, 28 \\ 18 \,.\, 27 \end{matrix}\right\}.$$

The last three forms of C are the transforms of the first by

$$\alpha\beta \,.\, xy \,.\, \gamma\delta \,.\left\{\begin{matrix} 56 \,.\, 78 \\ 57 \,.\, 68 \\ 58 \,.\, 67 \end{matrix}\right\}$$

respectively, and these substitutions transform J into itself. Similarly the groups generated by J and

$$\alpha x \,.\, 15 \,.\left\{\begin{matrix} \beta y \,.\, 26 \\ \gamma y \,.\, 37 \\ \delta y \,.\, 48 \end{matrix}\right\}$$

are conjugate under $\beta\gamma\delta \,.\, 678 \,.\, 234$. Then $K = \{J, \alpha x \,.\, \beta y \,.\, 15 \,.\, 26\}$. Here the systems of imprimitivity can be chosen in only one way, so that K is contained in a doubly transitive group of degree 15. Then we can find a substitution D similar to A introducing one letter θ new to K. Since K, J, I are all transitive, we can write $D = \theta x \,.\, \alpha\omega_1 \,.\, 1\omega_2 \,.\, \omega_3\omega_4$. Comparing D with I we see that ω_1 can not be one of the letter of I, and that ω_2 must be displaced by I. One of the substitutions of I contains the cycle $1\omega_2$ and must be commutative with D. If ω_3 and ω_4 are not also elements of I, some commutator $(A_1D)^2$ is of class equal to or less than 6. Then $\omega_2\omega_4$ is also a cycle of $1\omega_2 \ldots$ of I. Now comparing D with the substitutions of I', the transform of I by $5\alpha \,.\, 6\beta \,.\, 7\gamma \,.\, 8\delta$, it becomes evident that $\omega_1 = \beta$, and $\omega_2 = 2$, or $\omega_1 = \gamma$ and $\omega_2 = 3$, or $\omega_1 = \delta$ and $\omega_2 = 4$. Comparing D with I'', the transform of I by $1\alpha \,.\, 2\beta \,.\, 3\gamma \,.\, 4\delta$, we see that ω_3 and ω_4 are chosen from the elements 5, 6, 7, 8. If $\omega_1 = \beta$, the commutator of $D = \theta x \,.\, 12 \,.\, \alpha\beta \,.\, \omega_3\omega_4$ and $CBA_1BC = 12 \,.\, 78 \,.\, \gamma\delta \,.\, xy$ is of class less than 8. If $D = \theta x \,.\, \alpha\gamma \,.\, 13 \,.\, 68$ or $= \theta x \,.\, \alpha\delta \,.\, 14 \,.\, 67$, $(DCBA_2BC)^6$ is of degree 4. There remain only two possible forms for D:

$$\theta x \,.\, \alpha\gamma \,.\, 13 \,.\, 57 \quad \text{and} \quad \theta x \,.\, \alpha\delta \,.\, 14 \,.\, 58.$$

These two substitutions are conjugate under $\gamma\delta \,.\, 34 \,.\, 78$, a substitution which transforms the group K into itself. Hence we can assume $D = \theta x \,.\, \alpha\gamma \,.\, 13 \,.\, 57$ and the group $G = \{K, D\}$ is unique. The order of G is at least $8\,!/2$. The Abelian group of order 16 and type (1, 1, 1, 1) generated by

$$12 \,.\, 34 \,.\, 56 \,.\, 78 \,.\, \alpha\beta \,.\, \gamma\delta \,.\, xy \,.\, \theta\eta, \quad 13 \,.\, 24 \,.\, 57 \,.\, 68 \,.\, \alpha\gamma \,.\, \beta\delta \,.\, x\eta \,.\, y\theta,$$
$$15 \,.\, 26 \,.\, 37 \,.\, 48 \,.\, \alpha\eta \,.\, \beta\theta \,.\, \gamma x \,.\, \delta y, \quad 1\alpha \,.\, 2\beta \,.\, 3\gamma \,.\, 4\delta \,.\, 5\eta \,.\, 6\alpha \,.\, 7x \,.\, 8y$$

is transformed into itself by G, and hence G is the group of isomorphisms of this Abelian group, whose order is $8!/2$. This group is known to be isomorphic to the alternating group on 8 letters.

If G were contained in a larger primitive group G' of degree 15, G'_1, the subgroup of G' leaving θ fixed would contain G_1 self-conjugately. The subgroup G'_1 must be imprimitive on 7 systems which it permutes according to a group of class 4, that is, the $G^{7,4}_{168}$. Then G'_1 contains a new substitution S of order 2 transforming each system into itself. The intransitive head F of K contains two substitutions on 4 systems having in common any two arbitrary systems. If S is of degree 8 or 10 we can find two substitutions in F having two cycles in common with it, and hence one of them has three in common with S, so that when multiplied into S it gives a product of class 4 or 6. If S has 6 or 7 cycles, F contains a substitution having 4 cycles in common with it. Then G'_1 coincides with G_1 and G' with G.

We now pass to a G of degree 16. Recalling the determination of D, it is clear that E can be put uniquely in the form $\eta x \,.\, \alpha\delta \,.\, 14 \,.\, 58$. The conjugates of $(34)(78)(\alpha\beta)(xy) \times E$ generate the regular group of order 16 all of whose operators are of order 2, and hence the new group is the holomorph of this regular Abelian group.

3. *The* $D^{9,8}_{18}$.

In this and all the following cases no two substitutions of order 2 can displace exactly the same 8 letters.

Since the group of isomorphisms of the cyclic group of order 9 is the cyclic group of order 6 and class 6, this DG^{9}_{18} can not be contained in a larger group of the same degree and class 8.

Since the DG^{9}_{18} is only one-fold imprimitive, it is contained in a doubly transitive group of degree 10 and order 180, if contained in a primitive group at all. There are at least 9 conjugate subgroups of order 5 in G^{10}_{180}, and hence by Sylow's theorem as many as 36. This is impossible. This case gives no group.

its three subgroups of order 720 includes the $+\ G^{9,8}_{72}$. If a transitive group of degree 11 and order 7920 exists, it is the 4-ply transitive group of Mathieu and leads in turn to the 5-ply transitive group of order 12 . 11 . 10 . 8 . 9.*

We now return to the fundamental group of order 18. Let $A_1, \ldots, A_9$ be its substitution of order 2 in order as written. It may be assumed that there is a substitution B similar to A_1, connecting some new elements with the elements of I, and such that $\{B, I\}$ does not include one of the primitive groups we have already.

If B introduces one new letter, $\{I, B_1\}$ is of degree 10 and gives nothing new. The same is true if B brings in just two new letters. If B connects 3 new letters α, β, γ with the old letters of I, we must write $B = 1\alpha \,.\, \omega_1\beta \,.\, \omega_2\gamma \,.\, \omega_3\omega_4$. Since A_1 leaves the element 1 fixed, $BA_1B = \beta\gamma \ldots$, so that $\omega_1\omega_2$ is a cycle of A_1, and ω_3 and ω_4 are in different cycles of A_1. The subgroup of $G^{9,6}_{432}$ leaving the element 1 fixed is the group of isomorphisms of G^9_9 of type (1, 1) and in it A_1 is invariant. Then I is invariant under this G^8_{48}, leaving 1 fixed. The cycles of A_1 are the systems of imprimitivity of G_{48}. Further, the subgroup of G_{48} leaving the system 23 fixed is transitive. Then B can be written $1\alpha \,.\, 2\beta \,.\, 3\gamma \,.\, 4\left\{\begin{smallmatrix}5\\6\end{smallmatrix}\right\}$. Again, the substitution $23 \,.\, 56 \,.\, 89 \,.\, \beta\gamma$ transforms I into itself and finally fixes the substitution $B = 1\alpha \,.\, 2\beta \,.\, 3\gamma \,.\, 45$. The group $J = \{I, B\}$ has 4 systems of imprimitivity which are permuted according to a transitive group of class 2 containing a substitution of order 3; that is, according to the symmetric group on 4 elements. Since $(A_1B)^2 = 23 \,.\, 45 \,.\, 79 \,.\, \beta\gamma$ does not permute the systems, the subgroup which leaves all the systems fixed is of order 18 and J is of order 432. We pass from J to a group of degree 13. Since the systems of J can be chosen in only one way, it is contained in a doubly transitive group on 13 letters. Since in this $G^{13,8}_{5616}$ there are just 13 conjugate subgroups of order 18 similar to I, each subgroup is transformed into itself by a group of order 432, having one transitive constituent of degree 4 and order 24. The other constituent is the largest group on 9 letters which transforms I into itself. Hence $G^{13,8}_{5616}$ is completely determined by its subgroup leaving one letter fixed. This group of order 5616 is the doubly transitive group according to which the 13 subgroups of order 3 in the Abelian G_{27} of type (1, 1, 1) are permuted by its group of isomorphisms. This case is completed.

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* MATHIEU: *Liouville's Journal*, Ser. 2, Vol. VI (1861), p. 274. JORDAN: *Liouville's Journal*, Ser. 2, Vol. XVII (1872), p. 351.

5. *The Icosaedral* $G_{60}^{10,8}$.

This fundamental group is primitive. It can not be contained in a larger group of the same degree and class. It leads to a doubly transitive group on 11 letters, simply isomorphic to the modular group $G_{660}^{12,10}$. There is but one such group. Since in a $G_{7920}^{12,8}$ in which $G_{600}^{10,8}$ is maximal $G_{60}^{10,8}$ must be transformed into itself by 120 substitutions, the triply transitive group of order 7920 is unique. It is isomorphic to the 4-ply transitive group of degree 11.

6. *The Imprimitive* G_{120}^{12}.

This group is (2, 1)-isomorphic to $(abcde)+$. Suppose G_{120}^{12} to be included in an imprimitive G of degree 12. The presence of substitutions of degree 10 and order 5 requires that there be either 2 or 6 letters in each system of imprimitivity of the larger group. If there are just two systems, G has an intransitive subgroup of half its order. Then the letters of G_{120}^{12} can be arranged in two systems of 6 each, such that an intransitive head of order 60 is included in it. This head is a simple isomorphism between two icosaedral groups. In this head are included the 15 substitutions of class 8 of G_{120}^{12}, which contradicts the fact that the substitutions of class 8 generate the whole group. Now G_{120}^{12} can be included invariantly in a G_{240}^{12} in which the systems are permuted according to a symmetric group of order 120 written on 6 elements. But G_{120}^{12} can not be contained in a primitive group G of degree 12. For G can not be doubly transitive, nor, as we shall show, can it be simply transitive. In the latter case G_1 would be a simple isomorphism between two transitive groups of degrees 5 and 6 respectively. Now the only transitive groups of degree 5 which can be represented transitively on 6 letters include the icosaedral. This isomorphism is of class 6. We remark that the systems of imprimitivity of G_{120}^{12} can be chosen in only one way. Then G_{120} may be contained in a primitive group of degree 13. In this larger group a particular subgroup of order 2 and degree 8 is transformed into itself by 40 or 80 operators. But the largest group on the same letters transforming this substitution into itself is of order $2^4 . 4!$ and includes no operator of order 5.

7. *The Primitive* G_{720}^{15}.

We now consider the primitive $G_{720}^{15,8}$. The subgroup G_1 leaving one letter fixed is of order 48 and is a (2, 1) isomorphism between two transitive constituents of degrees 8 and 6 respectively. If G is invariant in a larger simply transitive group G' of degree 15, G_1 is invariant in G'_1, and the latter has the same sets of intransitivity as G_1. Now the head of the first (imprimitive) constituent in G_1 is the symmetric group of order 24, a complete group, so that any group in which it is invariant is a direct product. But the tail of this constituent already contains all the substitutions commutative with each operator of the head. Nor can G_{720} be invariant in a doubly transitive group of degree 15, for the only other substitution of class 8 we can determine is $B = 18 \,.\, 26 \,.\, 35 \,.\, ex$, bringing in one new letter x. Now $\{G, B\}$ has a regular invariant subgroup of order 16, and is of order 11,520. Any other substitution of class 8 we may determine leads again to B.

8. *The Primitive* $G^{16,8}_{1920}$.

If another substitution of class 8 and order 2 displacing no new letters occurs in a larger group containing $G^{16,8}_{1920}$, we shall have either $12 \ldots.$ or $1a \ldots.$ If it is $12 \ldots.$, we continue and see that it is $12 \,.\, 68 \,.\, 79 \ldots.$, an impossibility. If it is $1a \ldots.$, we get uniquely $1a \,.\, 6d \,.\, 7e \,.\, 5c$. The substitution $3a \,.\, 6b \,.\, ae \,.\, 8c$ transforms $\{G^{16}_{1920},\ 1a \,.\, 6d \,.\, 7e \,.\, 5c\}$ into $G^{16,8}_{11520}$.

Since the subgroup G_1 of G_{1920} is a complete group, G can not be invariant in a larger primitive group.

9. *The Imprimitive* $H^{16,8}_{576}$.

The fundamental transitive subgroup in this case is the imprimitive direct product of order $(24)^2$. Suppose this group is invariant in a primitive group G. Now H has a characteristic regular subgroup of order 16 and H_1 has a characteristic subgroup of order 9. Since the non-cyclic group of order 9 admits no isomorphism of period 5 or 7, G contains no operator of order 5 or 7. Then G_1 is of order $36 \,.\, 2$ or $36 \,.\, 4$ and can not be transitive. The transitive constituent of degree 9 in H_1 must be transformed into itself by G_1. We determine uniquely a substitution $B = a_2 b_1 \,.\, c_1 \alpha_1 \,.\, c_2 \alpha_2 \,.\, d_1 \beta_1 \,.\, d_2 \beta_2 \,.\, \gamma_2 \delta$, which gives us the primitive $G^{16,8}_{1152}$. The group G_{1152} is not in turn invariant in a larger group. If on the other hand H_{576} is not invariant in a larger group, there must be another substitution of class 8 and order 2. It is uniquely determined as

$$B = a_1 d_2 \,.\, a_2 d_1 \,.\, \gamma_1 \beta_2 \,.\, \gamma_2 \alpha_2 .$$

This group $\{H, B\}$ is a conjugate of G^{16}_{11520}.

We have obtained the following 18 primitive groups of class 8:

$$
\begin{array}{llllll}
G^{9}_{36}, & G^{9}_{72}, & \pm G^{9}_{72}, & G^{10}_{60}, & G^{10}_{360}, & \pm G^{10}_{720}, \\
G^{10}_{720}, & G^{11}_{660}, & G^{11}_{7920}, & G^{12}_{7920}, & G^{12}_{12.7920}, & G^{13}_{5616}, \\
G^{15}_{720}, & G^{15}_{8!/2}, & G^{16}_{1152}, & G^{16}_{1920}, & G^{16}_{11520}, & G^{16}_{16(8!/2)}.
\end{array}
$$

In JORDAN'S enumeration of the primitive groups of class less than 14, he mentions the groups of degree $\frac{K(K-1)}{2}$ formed by the displacements to which the symmetric (alternating) groups on K letters $a, b, c \ldots$ subject the $\frac{K(K-1)}{2}$ binary products $ab, ac, \ldots$. He states that these groups belong to the class $2K-2$ (to the class $3K-3$), and are primitive if $K > 4$. These formulae for the class are incorrect. For this representation of the symmetric group the class is $2K-4$, and for the alternating $4K-12$ when $K \leqq 6$, $3K-6$ when $K \geqq 6$.

A consequence of these formulae is that a simply transitive primitive group can always be found for which the ratio of the degree to the class is greater than any given number.

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Minimalcurven als Örter von Krümmungsmittelpunkten.

Von E. Study.

Die vorliegende Mittheilung enthält eine Erzeugungsweise der krummen Minimallinien, die, obwohl sie eigentlich recht nahe liegt, bis jetzt nicht bemerkt worden zu sein scheint, und einige sich daran anschliessende Entwickelungen, bei denen es sich im Grunde um eine Berührungstransformation handelt.

Wir benutzen die Methode und die Bezeichnungen, die kürzlich in den *Transactions of the American Mathematical Society* entwickelt worden sind. (Vol. X, 1909, pp. 1–49, citiert mit *A. C.*, und Vol. XI, 1910, citiert mit *N. Gl.*)

§ 1. *Curven auf Minimalkegeln.*

Der Formel, die zur Darstellung des Ortes der Krümmungsmittelpunkte einer regulären Curve dient,

$$y \equiv x + R \,.\, \beta \qquad \{A.\ C.,\ \text{Nr. } 10\}, \tag{1}$$

lässt sich folgender Lehrsatz entnehmen:

Unter den unebenen regulären Curven haben die mit constantem Radius der Schmiegungskugel die charakteristische Eigenschaft, dass ihr Torsionswinkel in constantem Verhältnisse steht zum Bogenelemente des zugehörigen Ortes der Krümmungsmittelpunkte.

Aus (1) ergiebt sich nämlich, mit Hülfe der *Frenet*'schen Gleichungen (*A. C.*, Nr. 9),

$$\frac{dy}{ds} \equiv \frac{dR}{ds}\beta + \frac{R}{T}\gamma, \tag{2}$$

also

$$\sqrt{dy|dy} \equiv \sqrt{\frac{R^2}{T^2} + \left(\frac{dR}{ds}\right)^2} \,.\, ds \equiv \mathfrak{R} \,.\, \frac{ds}{T}, \tag{3}$$

woraus unmittelbar der Satz abgelesen wird. Die unebenen regulären Curven mit constantem Radius $\mathfrak{R}$ der Schmiegungskugel sind theils solche der constanten

Krümmung $\pm\frac{1}{R}=\pm\frac{1}{\mathfrak{R}}$, theils sphärische Curven. Im Falle $\mathfrak{R}=0$ aber ist nur der zweite Fall möglich. So sehen wir, dass auch folgender Lehrsatz gilt:

Die unebenen Curven auf Minimalkegeln sind dadurch gekennzeichnet, dass die zugehörigen Örter von Krümmungsmittelpunkten Minimalcurven sind.

Die Minimallinien, die man auf diese Art findet, können niemals gerade sein, andere Besonderheiten aber haben sie nicht:

Auf jedem Minimalkegel liegt eine einzige analytische Curve, die eine vorgeschriebene krumme Minimallinie zum Ort der Krümmungsmittelpunkte hat.

Hiervon kann man sich sehr leicht überzeugen. Es sei o der Scheitel des gegebenen Minimalkegels. Der zweite Kegel, der die Curve (y), Ort der Punkte y, aus o projiciert, wird dann für die zu construierende Curve (x) als Evolutenfläche zu fungieren haben. Die Punkte o, x, y müssen nun ein rechtwinkliges Dreieck bilden, mit dem rechten Winkel bei y, und zwar ein Dreieck, dessen Hypotenuse die Länge Null hat. Nennen wir z den Punkt in der Mitte zwischen o und x, so ist z Mittelpunkt eines Kreises durch o, x, y, der in die Minimalgerade durch o und x und eine zweite Minimalgerade durch o und y zerfällt. Diese zweite Gerade aber ist Tangente an die Curve (y) im Punkte y. Man kann also etwa so verfahren: Man suche den eigentlichen Schnittpunkt z der Tangente des Punktes y mit dem gegebenen Minimalkegel, und verdoppele sodann den Vector (Minimalvector) von o nach z. Der Endpunkt x des verdoppelten Vectors wird dann auf dem Minimalkegel eine Curve beschreiben, zu der die Gerade $\overline{oy}$ als Krümmungsaxe des Punktes x gehört; y wird der Fusspunkt des Lothes von x auf diese Gerade, und der Ort von y also Ort der Krümmungsmittelpunkte für die Curve (x). Man erhält hieraus sofort

$$x \equiv 2y - \frac{(y/y)}{(y/y')}\cdot y'. \tag{4}$$

Der in dieser Formel auftretende Nenner (y/y') kann nicht identisch verschwinden, da dann $(y/y)\equiv$ const. folgen würde, die als krumm vorausgesetzte Mimimalcurve also sphärisch sein müsste, was unmöglich ist.

$$R^2 \equiv -(y/y) \tag{5}$$

ist der zu x gehörige quadrierte Krümmungsradius. Hat man aus (4) den Ort von x gefunden, so ergiebt sich aus diesem wieder der Ort von y nach der Regel (1), also

$$y \equiv x + \frac{(x'/x')\{(x'/x')x'' - (x'/x'')x'\}}{(x'x''/x'x'')}. \tag{6}$$

Vorausgesetzt ist hier, dass der vorgeschriebene Minimalkegel vom Anfangspunkt der Coordinaten ausgeht.

§ 2. *Specielle Darstellung der betrachteten Curven.*

Wir nehmen jetzt an, dass die *Minimalcurve* gegeben und dargestellt sei mit Hülfe einer charakteristischen Function $f(s)$ {deren dritter Differentialquotient nicht identisch verschwindet}, und zwar in der vom Verfasser bevorzugten Form:

$$\left.\begin{aligned} y_1 &\equiv i \,.\left\{f - s\,.f_1 - \frac{1-s^2}{2}\,.f_2\right\},\\ y_2 &\equiv \left\{f - s\,.f_1 + \frac{1+s^2}{2}\,.f_2\right\},\\ y_3 &\equiv -i\,.\{f_1 - s f_2\}. \end{aligned}\right\} \tag{7}$$

Man findet dann

$$R^2 \equiv -(y/y) \equiv f_1^2 - 2ff_2, \quad (y/y'_s) \equiv f \cdot f_3, \tag{8}$$

und

$$\left.\begin{aligned} x_1 &\equiv -\frac{i}{2f}\cdot\left\{f_1^2 - (sf_1 - 2f)^2\right\},\\ x_2 &\equiv \frac{1}{2f}\cdot\left\{f_1^2 + (sf_1 - 2f)^2\right\},\\ x_3 &\equiv \frac{i}{2f}\cdot\left\{2f_1\,.(sf_1 - 2f)\right\}. \end{aligned}\right\} \tag{9}$$

Vergleicht man diese Formeln mit den zur Darstellung der Curven auf einem Minimalkegel dienenden (*N. Gl.*, § 1, Nr. 7):

$$x_1 \equiv -i\,\frac{1-S^2(P)}{2\,S'(P)}, \quad x_2 \equiv \frac{1+S^2(P)}{2\,S'(P)}, \quad x_3 \equiv i\,.\,\frac{2\,S(P)}{2S'(P)}, \tag{10}$$

so ergiebt sich

$$S \equiv s - 2\frac{f(s)}{f_1(s)}, \quad \frac{dS}{dP} \equiv \frac{f(s)}{f_1^2(s)}, \tag{11}$$

und

$$\frac{dP}{ds} \equiv \frac{2ff_2 - f_1^2}{f(s)}.$$

Man kann also zunächst durch eine Quadratur den natürlichen Parameter P der Curve (x) als Function von s bestimmen, und hierauf, mit Hülfe einer Elimination, die Function $S(P)$. Die zur natürlichen Gleichung der Curve (x) gehörige Function $\Phi(P)$ hat den Werth

$$\Phi(P) \equiv \frac{1}{R^2} \equiv \frac{1}{f_1^2 - 2ff_2}. \tag{12}$$

Zweitens gehen wir von der *Curve auf dem Minimalkegel* (Nr. 10) als von dem Gegebenen aus. Diese Curve müssen wir als uneben voraussetzen, wenn wir wieder eine Curve, und folglich eine Minimalcurve, als Ort ihrer Krümmungsmittelpunkte erhalten wollen. Es ist also (*A. C.*, S. 34) der Fall $\Phi(P) \equiv$ const. auszuschliessen, der sich ja auch bei der eben ausgeführten umgekehrten Rechnung nicht ergeben hatte (Nr. 12). Es ist mithin anzunehmen, dass

$$S' S' S'''' - 4 S' S'' S''' + 3 S'' S'' S'' \not\equiv 0 \tag{13}$$

ist, womit insbesondere auch das Bestehen der Ungleichheit $3 S'' S'' - 2 S' S''' \not\equiv 0$ gesichert ist.

Nunmehr findet sich

$$\left.\begin{aligned}
y_1 &\equiv -\frac{i}{2}\,\frac{(1 - SS)(S'' S'' - S' S''') + 2 S' S' (S' S' - S S'')}{S' (3 S'' S'' - 2 S' S''')},\\
y_2 &\equiv \frac{1}{2}\,\frac{(1 + SS)(S'' S'' - S' S''') - 2 S' S' (S' S' - S S'')}{S' (3 S'' S'' - 2 S' S''')},\\
y_3 &\equiv \frac{i}{2}\,\frac{2 S (S'' S'' - S' S''') + 2 S' S' S''}{S' (3 S'' S'' - 2 S' S''')}.
\end{aligned}\right\} \tag{14}$$

Dies ist wirklich die Parameterdarstellung einer Minimallinie, deren Tangente in y den ebenfalls bekannten Punkt $\frac{1}{2}x$ enthält. Nennt man ξ irgend einen nicht verschwindenden Vector, der zu dieser Verbindungslinie gehört, so wird $\xi_1 : \xi_2 : \xi_3 \equiv dy_1 : dy_2 : dy_3$, und dann kann, da die Gleichungen (7) die Proportion

$$dy_1 : dy_2 : dy_3 \equiv -i\,\frac{1 - s^2}{2} : \frac{1 + s^2}{2} : 2is$$

nach sich ziehen (*A. C.*, Nr. 44), der zur Curve y gehörige Parameter s aus den Gleichungen

$$s \equiv \frac{\xi_1 + i\xi_2}{\xi_3} \equiv -\frac{\xi_3}{\xi_1 - i\xi_2}$$

gefunden werden. Bei Durchführung der Rechnung wird man etwa $\xi_k = y_k - \frac{1}{2}x_k$ setzen; man erhält dann

$$s \equiv S - 2\,\frac{S' S'}{S''}. \tag{15}$$

Die Gleichungen (11) zeigen nunmehr, dass

$$f \equiv \frac{S' S' S'}{S'' S''}, \quad f_1 \equiv -\frac{S'}{S''}, \quad f_2 \equiv \frac{S'' S'' - S' S'''}{S' (3 S'' S'' - 2 S' S''')} \tag{16}$$

sein muss. In der That liefert die Substitution der Werthe (15) und (16) in die Gleichungen (7) das Gleichungssystem (14). Durch Elimination des Parameters P ergiebt sich schliesslich f als Function von s.

Die abgeleiteten Resultate lassen sich kurz so zusammenfassen:

Die krumme Minimallinie $f = f(s)$ ist Ort der Krümmungsmittelpunkte für die auf dem Minimalkegel

$$x_1^2 + x_2^2 + x_3^2 = 0$$

verlaufende unebene Curve $S \equiv S(P)$.

Ist die Function $f(s)$ gegeben — so dass $f'''(s) \not\equiv 0$ — so werden P und $S(P)$ aus den Gleichungen

$$P \equiv \int \frac{2 f f'' - f' f'}{f'} \, ds, \quad S \equiv \frac{s f' - 2 f}{f'}$$

bestimmt. Es ist dann, wenn S als Function von P betrachtet wird, der Differentialausdruck

$$S' S' S'''' - 4 S' S'' S''' + 3 S'' S'' S''$$

nicht identisch gleich Null.

Wird umgekehrt die Function $S(P)$ dieser Bedingung gemäss angenommen, so dienen zur Bestimmung von s und $f(s)$, also zur Bestimmung der Minimalcurve, die Gleichungen

$$s \equiv \frac{S S'' - 2 S' S'}{S''}, \quad f \equiv \frac{S' S' S'}{S'' S''}.$$

Es kann dann, wenn f als Function von s betrachtet wird, der dritte Differentialquotient $f'''(s)$ nicht identisch verschwinden.

§ 3. *Minimalcurven, die aus anderen durch Derivation abgeleitet sind.*

Nach der vom Verfasser entwickelten Theorie (*A. C.*, § 6) gehört zu jeder krummen Minimallinie ein *sphärisches Bild*, ein Ort von Punkten auf einem beliebigen Minimalkegel. Dieses Bild kann aus einer einzigen analytischen Curve oder auch aus deren zweien bestehen, die dann durch die Spiegelung am Scheitel des Minimalkegels in einander übergehen. Der erste Fall tritt dann ein, wenn die beiden Werthe des Differentials des natürlichen Parameters auf der Curve einen analytischen Zusammenhang haben; der zweite, wenn dies nicht stattfindet. Entscheidet man sich im zweiten Falle für eine der beiden möglichen Annahmen über den natürlichen Parameter, so erhält man auch im zweiten Falle eine bestimmte analytische Curve als sphärisches Bild. Im ersten Falle aber wird man sich die Minimalcurve doppelt überdeckt denken, entsprechend den beiden Werthen des genannten Differentials. Sagen wir, die Curve sei durch die beschriebenen Processe *orientiert*, so entspricht nunmehr jedem Punkte der orientierten (einfach oder doppelt überdeckten) Curve eine völlig *bestimmte* Stelle des sphärischen Bildes, das jetzt immer eine einzelne analytische Curve ist.

Wir wollen nun die im vorigen Paragraphen betrachtete Curve auf dem Kegel $x_1^2 + x_2^2 + x_3^2 = 0$ mit dem sphärischen Bilde einer orientierten Minimallinie identificieren. Dies erreichen wir, wenn wir die letzte Curve in der Form

$$\left.\begin{aligned} Y_1 &\equiv i\left\{F - S F_1 - \frac{1-S^2}{2} F_2\right\}, \\ Y_2 &\equiv \left\{F - S F_1 + \frac{1+S^2}{2} F_2\right\}, \\ Y_3 &\equiv -i\left\{F_1 - S F_2\right\} \quad \{F_3 \not\equiv 0\} \end{aligned}\right\} \tag{17}$$

darstellen, über den Werth der Wurzelgrösse $\sqrt{F_3(s)}$ bestimmt verfügen, und den natürlichen Parameter P aus der Gleichung

$$P \equiv \int \sqrt{F_3}\,.\,dS \tag{18}$$

bestimmen. P ist dann zugleich natürlicher Parameter für das sphärische Bild der orientierten Curve:

$$x_1 \equiv -i\,\frac{1-S^2}{2}\sqrt{F_3}, \quad x_2 \equiv \frac{1+S^2}{2}\sqrt{F_3}, \quad x_3 \equiv i s \sqrt{F_3} \tag{19}$$

(vergl. *A. C.*, Nr. 48). Ist dieses Bild eine unebene Curve (*A. C.*, S. 34), gehört also die gegebene Minimalcurve nicht zur Familie der gemeinen Schraubenlinien (*A. C.*, S. 41), so gehört dazu eine Krümmungsmittelpunktscurve $f \equiv f(s)$ (Nr. 7) die als *Derivierte* der vorgelegten orientierten Minimalcurve $F \equiv F(S)$ (Nr. 17, 18) bezeichnet werden kann.

Die im vorigen Paragraphen entwickelten Formeln setzen nun den Zusammenhang in Evidenz, der zwischen den charakteristischen Functionen $F(S)$ und $f(s)$ der beiden betrachteten Minimalcurven besteht: Man hat nur in die Gleichungen (11, 15, 16) die Annahme $dP:dS \equiv \sqrt{F_3(S)}$ einzuführen. Es ergiebt sich also:

Zu jeder orientierten Minimalcurve $\{F \equiv F(S), \sqrt{F_3}\}$, *die nicht in der Familie der gemeinen Minimalschraubenlinien enthalten ist*

$$\{4 F_3^2 F_6 - 18 F_3 F_4 F_5 + 15 F_4^3 \not\equiv 0\},$$

gehört eine bestimmte krumme Minimallinie als Derivierte — Ort der Krümmungsmittelpunkte des sphärischen Bildes — die gefunden wird mit Hülfe der Gleichungen

$$s \equiv S + 4 \frac{F_3}{F_4}, \quad f(s) \equiv 4 \frac{F_3^2}{F_4^2} \sqrt{F_3}. \tag{20}$$

Umgekehrt ist jede krumme Minimallinie $\{f \equiv f(s), f_3(s) \not\equiv 0\}$ *Derivierte von* ∞^6 *orientierten Minimalcurven, die durch die Gruppe aller Schiebungen in einander übergehen. Diese Curven werden bestimmt aus den Gleichungen*

$$S \equiv s - \frac{2f}{f_1}, \quad \sqrt{F_3(S)} \equiv \frac{f_1^2}{f}. \tag{21}$$

Man überzeugt sich leicht durch Ausrechnung davon, dass jedes der Gleichungssysteme (20), (21) das andere zur Folge hat.

Das einfachste Beispiel zu unserem Satze liefern die Minimalcurven 3. Ordnung, betrachtet als Krümmungsmittelpunktscurven, wenn der Scheitel des Kegels $x_1^2 + x_2^2 + x_3^2 = 0$ auf der Curve selbst angenommen wird. Jede solche Figur ist congruent zu der durch die Annahme $f(s) = \frac{1}{3} s^3$ gegebenen. Sie gehört als Derivierte zu ∞^6 ebenfalls rationalen Minimalcurven 5. Ordnung.

NEAPEL, 5. *November*, 1909.

Minimalcurven und Serret'sche Flächen.

VON E. STUDY.

In seinem Aufsatz über die Differentialgeometrie der analytischen Curven (*Transactions of the American Mathematical Society*, Vol. X, 1909 — weiterhin mit *A. C.* citiert) hat der Verfasser gezeigt, dass bei jeder krummen Minimallinie die in der Formel

$$p \equiv \int \sqrt{\frac{(x' x'' x''')}{(x''|x'')}}\, dt$$

zusammengefassten Functionen Integralinvarianten sind, die sich ähnlicher Eigenschaften erfreuen, wie bei einer regulären Curve die Bogen genannten Integralinvarianten. Es ist zweckmässig, eine solche Function als Curvenparameter zu benutzen. Sie wird dann als ein *natürlicher Parameter* der Minimalcurve bezeichnet.

Es soll nunmehr nachgewiesen werden, dass dieser natürliche Parameter identisch ist mit der passend fixierten Bogenlänge gewisser Curven von der constanten Torsion Eins. Diese Curven sind, in ihrer Gesammtheit, mit der Minimalcurve invariant verbunden und punktweise auf sie abgebildet. Sie bilden die krummen asymptotischen Linien auf einer bestimmten Fläche constanter Krümmung, deren zweite Schaar asymptotischer Linien aus Minimalgeraden besteht. Man hat also, wie wir sagen werden, eine Röhrenfläche constanter Krümmung, eine sogenannte *Serret'sche Fläche* vor sich. Diese wird, nach einem Satze des Herrn *P. Stäckel*, durch eine zweite *Serret*'sche Fläche zur vollständigen Enveloppe einer Schaar congruenter Kugeln ergänzt, deren Mittelpunkte eben die betrachtete Minimalcurve erfüllen.* Für diese zweite Fläche findet sich, wie wir zeigen werden, die Torsion -1 der krummen asymptotischen Linien. Es gelingt aber auch — mit Hülfe des hier neu einzuführenden Begriffes *Quasidistanz* — die beiden Flächen im Voraus von einander zu trennen, so dass sie, bei gegebener Minimalcurve, jede für sich construiert werden können.

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* Die vollständige Enveloppe darf nach unserer Terminologie nicht Röhrenfläche genannt werden. Eine Fläche ist für uns, in der Differentialgeometrie, stets ein *analytisches* Gebilde.

In gleicher Weise gehören zu jeder krummen Minimallinie überhaupt unendlich viele *Serret*'sche Flächen, für deren einzelne man die (von Null verschiedene) Torsion der krummen asymptotischen Linien nach Belieben vorschreiben kann. Wir finden, dass diese unendlich vielen Flächen noch auf eine andere Art sich paaren lassen, so nämlich, dass den krummen asymptotischen Linien auf einer bestimmten Fläche eines Paares die krummen Minimallinien auf der anderen entsprechen. Hieraus ergiebt sich unmittelbar die isometrische Abbildung irgend einer *Serret*'schen Fläche auf eine Kugel.

Beiläufig entwickeln wir noch eine charakteristische Eigenschaft der Flächen von constanter mittlerer Krümmung.

§ 1. *Eine Eigenschaft der Curven constanter Torsion.*

Zu einer sehr einfachen geometrischen Deutung des natürlichen Parameters einer Minimalcurve kommt man auf Grund der folgenden Bemerkung:

Trägt man auf den Binormalen einer unebenen Curve von der constanten Torsion $\frac{1}{T}$ *eine Strecke von der constanten Länge* $\sqrt{-1}\,T$ *ab, so erhält man (irgend) eine krumme Minimallinie.*

Wir bezeichnen die Coordinaten eines Punktes X der Curve constanter Torsion mit X_1, X_2, X_3, und betrachten sie als Functionen des Bogens S der Curve. A, B, Γ seien die zur Tangente, Hauptnormale and Binormale der Curve gehörigen Einheitsvectoren, so dass $(\mathrm{A}\,\mathrm{B}\,\Gamma) \equiv 1$ wird. Die Behauptung ist dann zunächst, dass die Formel

$$x \equiv X + \sqrt{-1}\,.\,T\,.\,\Gamma, \tag{1}$$

bei bestimmtem Werthe der Wurzelgrösse $\sqrt{-1}$, eine Minimalcurve liefert. In der That ergeben die *Frenet*'schen Gleichungen

$$\frac{d\mathrm{A}}{dS} \equiv \frac{\mathrm{B}}{R}, \quad \frac{d\mathrm{B}}{dS} \equiv -\frac{\mathrm{A}}{R} + \frac{\Gamma}{T}, \quad \frac{d\Gamma}{dS} \equiv -\frac{\mathrm{B}}{T},$$

bei Differentiation nach dem Bogen S die Ausdrücke

$$\left.\begin{aligned} x' &\equiv \mathrm{A} - \sqrt{-1}\,\mathrm{B}, \quad x'' \equiv \frac{\sqrt{-1}}{R}\left\{\mathrm{A} - \sqrt{-1}\,\mathrm{B}\right\} - \frac{\sqrt{-1}}{T}\,.\,\Gamma,\\ x''' &\equiv -\frac{1 + \sqrt{-1}\,.\,R'}{R^2}\,.\left\{\mathrm{A} - \sqrt{-1}\,\mathrm{B}\right\} + \frac{\sqrt{-1}}{T^2}\,.\,\mathrm{B} + \frac{1}{RT}\,.\,\Gamma, \end{aligned}\right\} \tag{2}$$

aus deren erstem unmittelbar $(x'/x')\equiv 0$ folgt. Weiter aber folgt aus (2):

$$(x''/x'')\equiv -\frac{1}{T^2}, \quad (x'x''x''')\equiv -\frac{1}{T^3},$$

also — bei geeigneter Bestimmung einer Integrationsconstanten, deren Werth gleichgültig ist —

$$\left(\frac{dp}{ds}\right)^2 \equiv \frac{(x'x''x''')}{(x''/x'')} \equiv \frac{1}{T}, \quad p\equiv\frac{S}{\sqrt{T}}: \tag{3}$$

Der natürliche Parameter P der Minimalcurve (x) wird dem Bogen der Curve (X) proportional. Insbesondere folgt:

Ist $T\equiv 1$, *so kann der natürliche Parameter der Minimalcurve mit dem Bogen der Curve constanter Torsion identificiert werden.*

Wir berechnen noch die in der natürlichen Gleichung unserer Minimalcurve auftretende charakteristische Function

$$\Phi(p)\equiv(x_p'''/x_p''')\equiv\left(\frac{dS}{dp}\right)^6\cdot(x'''/x''')_S,$$

$$\Phi\equiv -T\left\{\frac{1}{T^2}+\frac{1}{R^2}-2\sqrt{-1}\,\frac{d}{dS}\left(\frac{1}{R}\right)\right\}. \tag{4}$$

Dies liefert, wenn Φ als Function von p bekannt ist, eine *Riccati*'sche Gleichung für das Krümmungsmass der Curve constanter Torsion,

$$\frac{d}{dp}\left(\frac{1}{R}\right)\equiv\frac{\sqrt{T}}{2\sqrt{-1}}\left\{\frac{1+T\Phi}{T^2}+\left(\frac{1}{R}\right)^2\right\}. \tag{5}$$

Hieraus folgt unmittelbar, dass die Aufgabe lösbar ist, Curven der Torsion $\frac{1}{T}$ zu finden, die auf die beschriebene Art eine *gegebene* krumme Minimallinie hervorbringen, und es zeigt sich, wie man alle diese Curven finden kann.

Sei nämlich $\frac{1}{R}$ eine *von Null verschiedene* Lösung der Gleichung (5), so betrachte man diese als Function der durch (3) erklärten Grösse S. $\frac{1}{R^2}$ und $\frac{1}{T}$ bestimmen dann, als Functionen von S, eine Classe unter einander congruenter regulärer Curven von der constanten Torsion $\frac{1}{T}$. Die Formel (1) liefert dann eine zugehörige Classe unter einander congruenter krummer Minimallinien. Diese gehören aber, nach (4), zur charakteristischen Function $\Phi\equiv\Phi(p)$; die gegebene Curve gehört daher selbst zur gefundenen Classe.

Sodann kann man, nach (2) und (3), den Einheitsvector Γ durch x_p' und x_p'' ausdrücken,

$$\Gamma \equiv \frac{\sqrt{T}}{R}\, x_p' - \frac{x_p''}{\sqrt{-1}}, \tag{6}$$

die Formel (1) liefert also

$$X \equiv x(p) - \sqrt{-1}\,\frac{T\sqrt{T}}{R} \,.\, x_p' + T \,.\, x_p''. \tag{7}$$

Jeder nicht identisch verschwindenden Lösung der Riccati'schen Gleichung (5) *entspricht also eine reguläre Curve der vorgeschriebenen constanten Torsion* $\frac{1}{T}$, *aus der auf dem beschriebenen Wege die gegebene krumme Minimallinie abgeleitet werden kann* $\left\{\frac{1}{T} \neq 0\right\}$.

Der Fall, der ausgeschlossen werden musste, kann nur dann eintreten, wenn Φ eine von Null verschiedene Constante und überdies

$$\frac{1}{T} = -\Phi$$

ist. Die Minimalcurve ist dann eine gemeine *transcendente* Schraubenlinie (*A. C.*, S. 37), die auf einem Rotationscylinder vom Radiusquadrate

$$-\frac{1}{\Phi^2}$$

geodätische Linie ist. Die Lösung Null der Gleichung (5) liefert dann nicht mehr eine reguläre Curve von der Torsion $\frac{1}{T}$, sondern *eine Euklidische Gerade,* nämlich die Axe des genannten Cylinders. Den übrigen Lösungen aber entsprechen auch in diesem Falle reguläre Curven.

Angemerkt zu werden verdient noch, unter welchen Umständen die Gleichung (1) eine oder zwei verschiedene Minimalcurven liefert. Es hängt das davon ab, ob man, durch Beschreibung eines geschlossenen Weges auf der Curve (X), die Orientierung der Binormale, oder den Einheitsvector Γ, umkehren kann oder nicht. Ausschlaggebend ist hierfür das Verhalten des Contingenzwinkels $dS:R$ (*A. C.*, Nr. 6, 8). Lässt sich dieser als einwerthige Function des Ortes auf der Curve (X) erklären, so erhält man zwei analytisch-getrennte Minimalcurven, andernfalls nur eine.

Bevor wir aus den abgeleiteten geometrischen Sätzen weitere Folgerungen ziehen, wollen wir noch einer besonderen Darstellung der krummen Minimallinien gedenken, die in unseren Formeln steckt.

Nach der Definition des Torsionsbegriffs ist der Bogen irgend einer unebenen Curve von constanter Torsion proportional zum entsprechenden Bogen ihrer Binormalenindicatrix. Hieraus ergiebt sich unmittelbar die bekannte Darstellung der Curven von constanter nicht verschwindender Torsion durch Quadraturen: Man verstehe unter $\Gamma_k(s) \equiv \xi_k(s)$ irgend drei Functionen (analytische Functionen) mit gemeinsamem Existenzbereich, die den Bedingungen

$$(\xi/\xi) \equiv 1, \quad (\xi'/\xi')_s \equiv 1 \tag{8}$$

$$(\xi\,\xi'\,\xi'')_s \not\equiv 0 \tag{9}$$

Genüge leisten. Dann liefert die Formel

$$X \equiv \int \xi\left(\frac{S}{T}\right) \wedge \xi'\left(\frac{S}{T}\right) dS \quad \left\{\frac{1}{T} \neq 0\right\} \tag{10}$$

alle Curven der vorgeschriebenen constanten Torsion $\frac{1}{T}$, bezogen auf deren Bogen $S \equiv Ts$ als Parameter.

Einer Erläuterung bedarf, wie es scheint, die Ungleichheit (9). Im Falle $(\xi\,\xi'\,\xi'')_s \equiv 0$ würden die Formeln (10) offenbar eine Gerade, oder ein Geradenstück liefern, also nicht eine Curve von constanter Torsion.* Es ist ja auch von vornherein klar, dass die Binormalen einer regulären Curve nur so zu einer festen Ebene parallel sein können, dass sie alle zu einander parallel sind, sowie dass in diesem Grenzfalle, der von den regulären ebenen Curven gebildet wird, die Binormalenindicatrix als Curve oder Curvenstück zu existieren aufhört.

Aus dem Gesagten ergiebt sich

$$x \equiv \sqrt{T} \int \xi\left(\frac{p}{\sqrt{T}}\right) \wedge \xi'\left(\frac{p}{\sqrt{T}}\right) dp + \sqrt{-1}\,.\,T\,.\,\xi\left(\frac{p}{\sqrt{T}}\right). \tag{11}$$

Für die so dargestellten Minimalcurven ist p natürlicher Parameter.

Die Formel (11) liefert aber krumme Minimallinien auch dann noch, wenn man die Einschränkung (9) fallen lässt. Da an Stelle der Curve constanter Torsion dann, wenn $(\xi\,\xi'\,\xi'') \equiv 0$ ist, eine Euklidische Gerade tritt, so hat man den schon genannten Fall der transcendenten gemeinen Minimalschraubenlinien vor sich.

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* Die hiernach unerlässliche Einschränkung (9) ist, wie es den Anschein hat, bisher immer übersehen worden. Siehe zum Beispiel *Darboux* (I, p. 42), *Bianchi* (S. 34), *Scheffers* (S. 252).

Alle krummen Minimallinien erhält man aus (11) bereits, wenn man $\sqrt{T}=1$ setzt. Gegenüber der früher von uns vorgetragenen hat diese viel elegantere Darstellung der krummen Minimallinien durch einen natürlichen Parameter den Nachtheil der Mehrdeutigkeit.

§ 2. *Röhrenflächen constanter Krümmung.*

Wir betrachten jetzt die Gesammtheit der Curven von der Torsion $\frac{1}{T}$, die zu einer bestimmten krummen Minimallinie gehören. Diese Curven werden auf einer gewissen Fläche liegen, und sie werden diese Fläche erfüllen, abgesehen möglicher Weise von Stellen besonderer Lage (wie sie in dem besprochenen Grenzfalle auftraten). Ferner sind sie offenbar *asymptotische Linien* auf der genannten Fläche. Diese hat daher die constante Krümmung $-\frac{1}{T^2}$. Der Punkt x ist aber, für reguläre Punkte X der Fläche, einer der Krümmungsmittelpunkte; und da das Quadrat des Abstandes zwischen x und X gleich $-T^2$ ist, so fällt der zweite Krümmungsmittelpunkt zu X in denselben Punkt x. Die zweite Schaar von asymptotischen Linien der gefundenen Fläche lässt sich ebenfalls angeben. Nach (7) schickt nämlich der Punkt x seine Schmiegungsebene durch den Punkt X.* Da das Quadrat des Abstandes beider Punkte lediglich von T abhängt, so beschreibt der Punkt X, wenn man, nach Anweisung der Gleichung (5), den Werth von R variiert, x aber festhält, eine gerade Linie, parallel zur Tangente der Minimalcurve in x.

Die gefundene Minimalgerade ist asymptotische Linie auf der betrachteten Fläche, und dann nothwendig zugleich, zweifach zählend, Krümmungslinie. Zwei bestimmte asymptotische Linien der zweiten Schaar werden dann, nach einem bekannten Satze, auf je zwei asymptotischen Linien der ersten Schaar Paare von Punkten bestimmen, zwischen denen (bei entsprechenden Integrationswegen) gleiche Bogenlängen liegen. Dies ist der von uns schon auf andere Art begründete Satz, dass die Bogenlänge zwischen zwei Punkten einer asymptotischen Linie der ersten Schaar nur abhängt von der Parameterdifferenz der entsprechenden Punkte der Minimalcurve, also nicht variiert, wenn man $\frac{1}{R}$ durch eine andere Lösung der Gleichung (5) ersetzt.

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* Die Curve (x) ist zugleich asymptotische und geodätische Linie auf der Binormalenfläche von (X).

Ferner ist klar, dass die betrachteten Flächen, für die sich aus (7) die Parameterdarstellung

$$X \equiv x(p) + Tx''(p) + x'(p) \cdot q \qquad (12)$$

ergiebt, zu zweien zusammengehören, da die Substitution von $-T$ an Stelle von T das Krümmungsmaass nicht ändert: Zwei solche Flächen werden dann zusammen die Enveloppe einer Schaar von Kugeln bilden, die ihre Mittelpunkte auf der gegebenen Minimalcurve haben, entsprechend dem Werthe $-T^2$ des quadrierten Radius.*

Wir nennen *Röhrenfläche* jede analytische Fläche, die Enveloppe oder Bestandtheil der Enveloppe einer Schaar congruenter (eigentlicher) Kugeln ist, deren Mittelpunkte eine analytische Curve erfüllen. Zu den Röhrenflächen gehören dann alle Minimalebenen, Minimalkegel, und Tangentenflächen von Minimalcurven, und von Flächen, auf die die gewöhnliche Theorie anwendbar ist, unter underen die, von denen wir soeben gesprochen haben.

Ist die erzeugende Kugel einer Röhrenfläche nicht zugleich Minimalkegel, so ist die Röhrenfläche die vollständige Enveloppe der Kugelschaar immer dann, wenn der Ort der Kugelmittelpunkte keine Minimalcurve ist. Ist dieser Ort aber eine krumme Minimallinie,† *so besteht die vollständige Enveloppe aus zwei geradlinigen Röhrenflächen, mit entgegengesetzt-gleicher (constanter) Torsion der krummen asymptotischen Linien.*

Alle Röhrenflächen von nicht verschwindender constanter Krümmung, und, abgesehen von den Kugeln, alle geradlinigen Flächen von nicht verschwindender constanter Krümmung werden auf diese Art mit Hülfe von Minimallinien construiert.

Um im Falle der krummen Minimallinien die beiden Bestandtheile der Enveloppe von einander trennen zu können, muss man von vorn herein sich für den einen der in jedem Falle möglichen beiden Werthe von T entscheiden.

Zwei in derselben, einem Minimalvector η ($\eta \neq 0$) zugehörigen Minimalebene gelegene Punkte x, X bestimmen nun *eindeutig* eine gewisse Bewegungsinvariante, die nicht auch (absolute) Umlegungsinvariante ist, und die, wenn

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* Fast alle die genannten Eigenschaften hat bereits Herr *P. Stäckel* angegeben (*Leipz. Ber.*, 1902, S. 168 u. ff.), wo man auch noch weitere Litteratur angeführt findet. Herr *Stäckel* macht jedoch keine Angabe über das Verhalten der Torsion der krummen asymptotischen Linien auf den gepaarten Flächen.

Wie man durch Construction der Minimalgeraden durch die einzelnen Curvenpunkte in den Schmiegungsebenen einer Curve constanter Torsion zu *Serret*'schen Flächen kommen kann, hat, nach mündlicher Mittheilung, bereits Herr *Bianchi* gezeigt. Offenbar kann man auch von hier aus die Sätze des Textes ohne Mühe entwickeln. Kennt man, wie bei der *Bianchi*'schen Construction, von vorn herein eine krumme asymptotische Linie auf einer *Serret*'schen Fläche, so verlangt die Bestimmung der übrigen natürlich nur Quadraturen.

† Eine Minimalgerade bestimmt keine Röhrenflächen.

$$\vartheta = X - x$$

gesetzt wird, in der Form

$$\frac{(\vartheta\,\eta\,\omega)}{(\eta/\omega)} \qquad \{(\eta/\omega) \neq 0\}$$

dargestellt werden kann (*A. C.*, S. 25, 26). Ihr Quadrat ist gleich dem negativen Quadrate des Abstandes der Punkte x, X $\{= -(\vartheta/\vartheta)\}$. Sie selbst ändert sich nicht, wenn man z. B. den Punkt X auf der zugehörigen zu η parallelen Minimalgeraden verschiebt. Wir können diese Grösse etwa die *Quasidistanz* von der Minimalgeraden durch x zu der parallelen Minimalgeraden durch X nennen.

Im vorliegenden Falle können wir, an allen regulären Stellen der Minimalcurve (x), $\eta \equiv x'$ setzen. Da bei Darstellung einer krummen Minimallinie durch ihren natürlichen Parameter

$$(x'\,x''\,\omega) \equiv -(x'/\omega)$$

ist, so liefert die Formel (7) den Werth der Quasidistanz

$$\frac{(\vartheta\,x'\,\omega)}{(x'/\omega)} \equiv -T\frac{(x'\,x''\,\omega)}{(x'/\omega)} \equiv T.$$

Für $T = 1$ ergiebt sich insbesondere:

Man suche in der Schmiegungsebene des Punktes x der gegebenen Minimalcurve die Minimalgerade, deren Quasidistanz von der Tangente des Punktes x gleich der Längeneinheit ist. Der geometrische Ort dieser geraden Linien ist eine Röhrenfläche von der Krümmung -1. Ihre krummen asymptotischen Linien haben die Torsion Eins. Der Bogen dieser asymptotischen Linien zwischen denselben zwei geraden Erzeugenden der Fläche ist bei entsprechenden Integrationswegen und geeigneter Bestimmung der vorkommenden Wurzelgrössen in allen Fällen derselbe, und gleich der Differenz der zugehörigen Werthe eines natürlichen Parameters der Minimalcurve.

Offenbar kann dieser Lehrsatz auch benutzt werden zur *Definition* des natürlichen Parameters p einer krummen Minimallinie, der hiermit eine, man möchte beinahe sagen anschauliche, geometrische Deutung erhält.

Die Darstellung der Röhrenflächen constanter Krümmung durch Formeln, die frei sind von Quadraturen, insbesondere die Darstellung aller algebraischen Flächen der Art, ist eine unmittelbare Folgerung unserer Theorie. Man leitet ohne Weiteres aus (12) die Parameterdarstellung her:

$$\left.\begin{aligned} X_1 &\equiv i\left\{f - s f_1 - \frac{1-s^2}{2} f_3 + T\,\frac{1-st}{s-t}\right\}, \\ X_2 &\equiv -\left\{-f + s f_1 - \frac{1+s^2}{2} f_3 + T\,\frac{1+st}{s-t}\right\}, \\ X_3 &\equiv -i\left\{f_1 - s f_3 + T\,\frac{s+t}{s-t}\right\} \qquad \{f_3(s) \not\equiv 0\} \end{aligned}\right\} \tag{13}$$

In der That ist die Gerade $s = \text{const.}$ parallel zur Tangente des Punktes $x(s)$, und für $t = \infty$ ergiebt sich offenbar ein Punkt auf der von uns construierten Parallelen, da x_p'' sich von dem Vector mit den Coordinaten is, s, i nur um ein Multiplum des Vectors x_p' unterscheidet. Das zu (13) gehörige quadrierte Bogenelement

$$(dX/dX) \equiv 2\,T . f_3(s) . ds^2 - \frac{4\,T^2}{(s-t)^2} . ds\,dt \tag{14}$$

hat wirklich das Krümmungsmaass $-\frac{1}{T^2}$.[*]

§ 3. *Isometrische Abbildung der betrachteten Flächen auf Kugeln.*

Die vorausgehenden Überlegungen führen unmittelbar zur Bestimmung der krummen Minimallinien auf den Röhrenflächen constanter nicht verschwindender Krümmung, und damit zu deren isometrischer Abbildung auf Kugelflächen. Indessen handelt es sich hierbei um einen Specialfall allgemeinerer Sätze, die wir vorausschicken wollen.

Wir betrachten zuerst die Beziehung zwischen Flächen constanter Krümmung und Flächen von constanter mittlerer Krümmung, die, dem Grundgedanken nach, von *Bonnet* angegeben worden ist. Der *Bonnet*'sche Lehrsatz lautet wie folgt:[†]

Trägt man auf den Normalen einer nicht geradlinigen Fläche der constanten nicht verschwindenden Krümmung K Strecken von den constanten Längen

$$\sqrt{-1}\,T = \frac{1}{\sqrt{K}}$$

ab, so sind die den verschiedenen Wurzelwerthen entsprechenden geometrischen Örter

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[*] Die Formeln (13) sind identisch mit solchen bei *Stäckel* (S. 116), abgesehen von der Bezeichnung und von der dort unbestimmt gelassenen Wurzelgrösse. Die von Herrn *Stäckel* mitgetheilte Formel für das Bogenelement und die daraus abgeleiteten Folgerungen sind indessen irrthümlich.

[†] Nach Verbesserung von einigen kleinen Ungenauigkeiten, die auch in diesem Falle wohl durch die ganze Literatur gehen.

ihrer Endpunkte zwei Parallelflächen zur gegebenen Fläche, mit constanten mittleren Krümmungen *

$$\sqrt{K} = -\frac{\sqrt{-1}}{T}.$$

Ist die Fläche constanter Krümmung dagegen geradlinig, ist sie also eine Röhrenfläche oder eine Kugel, so tritt im ersten Falle an Stelle einer der genannten Parallelflächen eine krumme Minimallinie, im zweiten der Mittelpunkt der Kugel.

Wir übergehen die evidente Umkehrung, und ziehen sogleich die Folgerung, auf die es uns hier ankommt:

Durch die Transformation des Bonnet'schen Satzes werden den asymptotischen Linien auf den vorausgesetzten Flächen constanter Krümmung zugeordnet Minimalcurven auf den entsprechenden Flächen constanter mittlerer Krümmung.

Erklären wir, bei gegebenem Integrationsweg, als *Quasidistanz* zweier Punkte auf einer krummen Minimallinie den entsprechenden Werthzuwuchs eines natürlichen Parameters, oder aber *die Null*, wenn die Minimallinie gerade ist, so folgt weiter:

Die Bogenlängen der asymptotischen Linien auf den Flächen der constanten nicht verschwindenden Krümmung $K = -\frac{1}{T^2}$ *gehen aus den entsprechenden Quasidistanzen auf den zugeordneten Flächen von constanter mittlerer Krümmung durch Multiplication mit* $\sqrt{T}$ *und* $\sqrt{-T}$ *hervor.*

Man kann sich auf der Fläche constanter mittlerer Krümmung ein Gebiet derart abgegrenzt denken, dass durch jeden Punkt im Inneren dieses Gebietes zwei in der Umgebung des Punktes getrennte Minimallinien gehen. Geeignet gewählte verschiedenartige Paare gleichartiger Minimalcurven bestimmen dann gewisse Figuren von vier Punkten, die wir "Elementarvierecke" auf der Fläche nennen wollen. Wir können dann weiter aussagen:

Auf den Flächen constanter mittlerer Krümmung haben in jedem Elementarviereck die durch gegenüberliegende Seiten (Minimallinien) verbundenen Ecken gleiche Quasidistanz.

Dies ist zwar durch das Vorhergehende nur begründet für Flächen von nicht verschwindender mittlerer Krümmung. Dass indessen für diese, also für die (regulären) Minimalflächen, der Satz in Geltung bleibt, ist evident.*

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* Wir folgen dem vorwiegenden Sprachgebrauch: Mittlere Krümmung = Summe (nicht halbe Summe) der reciproken Werthe der Hauptkrümmungsradien.

Die im letzten Satze genannte Eigenschaft ist für die Flächen von constanter mittlerer Krümmung charakteristisch.

Es sei $x \equiv x(s, t)$ irgend eine Fläche, für die der Begriff der mittleren Krümmung überhaupt Sinn hat, nämlich eine solche, die weder Minimalebene, noch Minimalkegel, noch Tangentenfläche einer Minimalcurve ist:

$$(x_s x_t / x_s x_t) \not\equiv 0.$$

Die Coefficienten der ersten und zweiten zugehörigen quadratischen Differentialform

$$E\,ds^2 + 2F\,ds\,dt + G\,dt^2, \quad e\,ds^2 + 2f\,ds\,dt + g\,dt^2$$

werden dann, wenn

$$D \equiv \sqrt{EG - F^2} \equiv \sqrt{x_s x_t / x_s x_t}$$

gesetzt wird, durch die Ausdrücke

$$E \equiv (x_s/x_s), \quad F \equiv (x_s/x_t), \quad G \equiv (x_t/x_t),$$
$$D.e \equiv (x_s x_t x_{ss}), \quad D.f \equiv (x_s x_t x_{st}), \quad D.g \equiv (x_s x_t x_{tt})$$

gegeben. Wir nehmen nun Minimallinien als Coordinatenlinien an, so dass

$$E \equiv 0, \quad G \equiv 0, \quad D \equiv \sqrt{-1}\,.\,F$$

wird. Es folgt dann

$$D\,.\,(x_{ss}/\omega) \equiv e\,.\,(x_s x_t \omega) + \sqrt{-1}\,F_s\,(x_s/\omega),$$
$$D\,.\,(x_{st}/\omega) \equiv f\,.\,(x_s x_t \omega),$$
$$D\,.\,(x_{tt}/\omega) \equiv g\,.\,(x_s x_t \omega) + \sqrt{-1}\,F_t\,(x_t/\omega),$$

und, nach kurzer Rechnung,

$$(x_{ss}/x_{ss}) \equiv e^2, \quad (x_s x_{ss} x_{sss}) \equiv \sqrt{-1}\,.\,e^3,$$
$$(x_{tt}/x_{tt}) \equiv g^2, \quad (x_t x_{tt} x_{ttt}) \equiv -\sqrt{-1}\,.\,g^3.$$

Die natürlichen Parameter der Coordinatenlinien werden also zu berechnen sein aus den Gleichungen

$$\left(\frac{dp}{ds}\right)^2 \equiv \sqrt{-1}\,.\,e\ \{t = \text{const.}\}, \quad \left(\frac{dq}{dt}\right)^2 \equiv -\sqrt{-1}\,.\,g\ \{s = \text{const.}\}.$$

Unsere Voraussetzung besagt daher, dass e frei ist von t und g frei von s.

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* Der Verfasser hat sich überzeugt, dass durch diesen Satz eine nicht unbedeutende Vereinfachung der Theorie der Minimalflächen bewirkt wird, und dass sich daraus auch manches Neue ergiebt. Wir behalten uns vor, hierauf in einer besonderen Abhandlung näher einzugehen.

Die *Codazzi*'schen Gleichungen lehren nunmehr, dass f sich von F nur um einen constanten Zahlenfactor unterscheidet (der im Falle der Minimalflächen gleich Null ist). Es ergiebt sich also für die mittlere Krümmung $2f : F$ ein constanter Werth.

Verschwindet weder e noch g identisch, so kann man p und q selbst als Parameter einführen. Die Fläche ist dann nicht geradlinig. Verschwindet e *oder* g identisch, so hat man eine unserer geradlinigen Röhrenflächen vor sich. Sind e und g beide identisch gleich Null, so ist die Fläche eine Kugel oder eine Euklidische Ebene.

Aus dem soeben Vorgetragenen, aber auch schon aus den Betrachtungen des vorigen Paragraphen ergiebt sich nun folgender Satz:

Kennt man bei einer Röhrenfläche von nicht verschwindendem constantem Krümmungsmass die krummen asymptotischen Linien, so kennt man bei einer zweiten zur ersten parallelen Fläche der Art die krummen Minimallinien, und umgekehrt. Die Torsionen der krummen asymptotischen Linien so gepaarter Flächen verhalten sich wie 2 : 1.

Wir betrachten die erste zur Torsion $1 : T$ der asymptotischen Linien gehörige Fläche als Fläche der constanten Krümmung $-\frac{1}{T^2}$, die zweite als Fläche der constanten mittleren Krümmung $-\frac{\sqrt{-1}}{T}$. Wir berechnen, wie zuvor, den Einheitsvector Γ (Nr. 6) aus der *Riccati*'schen Gleichung (Nr. 5), ersetzen aber die Gleichung (1),

$$X \equiv x - \sqrt{-1} \,.\, T \,.\, \Gamma,$$

durch die andere

$$Y \equiv x - 2\sqrt{-1} \,.\, T \,.\, \Gamma. \tag{15}$$

Lässt man dann die Lösung der Gleichung (5), entsprechend den Werthen eines geeigneten Parameters q variieren, so wird der Ort der gefundenen Curven

$$Y \equiv x(p) - \frac{2\sqrt{-1}\, T \sqrt{T}}{R(p, q)}\, x'(p) + 2\, T x''(p) \tag{16}$$

eine dem Torsionswerthe $1 : 2T$ entsprechende geradlinige Röhrenfläche, *die auf Minimalcurven als Parameterlinien bezogen ist.*

In der That erhält man unter der Annahme

$$\frac{\partial}{\partial p}\left(\frac{1}{R}\right) \equiv \frac{\sqrt{T}}{2\sqrt{-1}} \left\{\frac{1 + T\Phi(p)}{T^2} + \left(\frac{1}{R(p, q)}\right)^2\right\} \tag{17}$$

für das quadrierte Bogenelement der Fläche (16) den Ausdruck

$$(dY/dY) \equiv -4\,T^2 \,.\, 2\sqrt{-1}\,\sqrt{T}\,\frac{\partial}{\partial q}\left(\frac{1}{R}\right) dp\,dq, \tag{18}$$

dem das constante Krümmungsmaass $-\dfrac{1}{4\,T^2}$ zukommt.

Natürlich wird bei ungeschickter Wahl des Parameters q die Gleichung (16) nicht die ganze Fläche darstellen, um die es sich handelt, sondern nur ein Stück davon. Dies kann man aber immer vermeiden, zum Beispiel indem man $\dfrac{1}{R}$ als lineare (in der Regel gebrochene) Function von q annimmt, was nach der allgemeinen Theorie der *Riccati*'schen Gleichungen statthaft ist. Man braucht dazu nur drei Particularlösungen der Gleichung (5) zu kennen.

Es handelt sich schliesslich noch darum, die quadratische Differentialform

$$2\sqrt{-1}\,\sqrt{T}\,\frac{\partial}{\partial q}\left(\frac{1}{R}\right) dp\,dq \tag{19}$$

vom Krümmungsmaasse Eins in die Gestalt

$$\frac{4\,d\sigma\,d\tau}{(\sigma-\tau)^2}$$

überzuführen. Die Theorie dieses Problems ist bekannt; sie braucht uns hier nicht weiter zu beschäftigen.*

Dass die vorgetragene Methode nicht daran haftet, dass gerade der natürliche Parameter der benutzten Minimalcurve als unabhängige Veränderliche dient, ist leicht zu sehen. Die zur Bestimmung dieses Parameters in der Regel erforderliche Quadratur (und Elimination) lässt sich ersparen. *Zur Ausführung der isometrischen Abbildung der Fläche* (Y) *auf eine Kugel vom Radius* $2\sqrt{-1}\,T$ *ist nur erforderlich die Lösung einer Riccati'schen Gleichung* (keine Elimination).

§ 4. *Beispiele.*

Die ganze zuvor entwickelte Theorie lässt sich bis zu geschlossenen Endformeln durchführen in den Fällen, in denen die gegebene Minimalcurve zur Familie der gemeinen Schraubenlinien gehört $\{\phi \equiv \text{const.}\}$. Die zugehörigen Röhrenflächen sind dann Schraubenflächen mit Minimalgeraden als Erzeugenden.

Wir gehen aus von irgend einer gemeinen transcendenten Minimalschraubenlinie, deren Parameterdarstellung wir in der Form

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* S. *Darboux*, Théorie des surfaces, I, Nr. 52.

$$x_1 \equiv \sqrt{ir}.p, \quad x_2 \equiv r\cos\frac{p}{\sqrt{ir}}, \quad x_3 \equiv r\sin\frac{p}{\sqrt{ir}}$$

annehmen dürfen. Es folgt dann

$$\Phi(p) \equiv (x'''/x''')_p \equiv \frac{i}{r}.$$

Sei zuerst

$$T \not\equiv ir,$$

so ergiebt sich als Lösung der Gleichung (17), wenn zur Abkürzung

$$C = -\frac{r+iT}{4rT}$$

gesetzt wird,

$$\frac{1}{R} \equiv 2\sqrt{-1}.\frac{\sqrt{C}}{\sqrt{T}}\,tg\,\{\sqrt{C}.(p-q)\}.$$

Dabei kann der mit $\sqrt{-1}$ bezeichneten Wurzelgrösse nach Belieben der Werth i oder der Werth $-i$ beigelegt werden. Tragen wir den gefundenen Werth von r ein in die Formel

$$X(p,q) \equiv x(p) - \sqrt{-1}.\frac{T\sqrt{T}}{R(p,q)}.x'(p) + T.x''(p), \tag{20}$$

so haben wir damit die zur Torsion $\frac{1}{T}$ gehörigen Röhrenflächen bestimmt und bezogen auf ihre *asymptotischen Linien.* Gleichzeitig werden bei Substitution desselben Werthes die zur Torsion $\frac{1}{2T}$ gehörigen Röhrenflächen auf ihre *Minimalcurven* bezogen, wenn man

$$Y(p,q) \equiv x(p) - 2\sqrt{-1}.\frac{T\sqrt{T}}{R(p,q)}.x'(p) + 2T.x''(p) \tag{21}$$

setzt.

Für das quadrierte Bogenelement der zweiten Fläche erhält man den Ausdruck

$$(dY/dY) \equiv -4T^2.\frac{4C\,dp\,dq}{\cos^2\{\sqrt{C}(p-q)\}}, \tag{22}$$

dessen zweiter Factor das Krümmungsmaas Eins hat. Setzt man nun

$$\sigma(p) \equiv tg\,\{\sqrt{C}.p\}, \quad \tau(q) \equiv -ctg\,\{\sqrt{C}.q\},$$

so wird

$$(dY/dY) \equiv -4T^2.\frac{4\,d\sigma\,d\tau}{(\sigma-\tau)^2}. \tag{23}$$

Hiermit ist also die Fläche isometrisch bezogen auf eine Kugel, deren Parameterdarstellung

$$\xi_1 \equiv 2i\,T.\frac{1-\sigma\tau}{\sigma-\tau}, \quad \xi_2 \equiv -2\,T.\frac{1+\sigma\tau}{\sigma-\tau}, \quad \xi_3 \equiv -2i\,T.\frac{\sigma+\tau}{\sigma-\tau}$$

ist.

Viel einfacher noch verhält sich der Grenzfall

$$T = ir,$$

in dem die erste der construierten Flächen die Axe der Schraubenlinie als ausgezeichnete asymptotische Linie enthält. Man hat dann

$$\frac{1}{R} \equiv -\frac{2\sqrt{-1}}{\sqrt{T}(p-q)},$$

$$(dY/dY) \equiv -4\,T^2.\frac{4\,dp\,dq}{(p-q)^2}.$$

Hier vermittelt also schon die Substitution

$$\sigma \equiv p, \quad \tau \equiv q$$

die isometrische Abbildung der Fläche $Y(p,q)$ auf eine Kugelfläche.

In gleicher Weise behandelt man die algebraischen Grenzfälle der betrachteten transcendenten Schraubenröhrenflächen, die sämmtlich erhalten werden, wenn man etwa von der Minimalcurve 3. Ordnung

$$x_1 \equiv i\left\{\frac{p^3}{6} - \frac{1}{2}p\right\}, \quad x_2 \equiv \frac{p^3}{6} + \frac{1}{2}p, \quad x_3 \equiv -\frac{i}{2}p^2$$

ausgeht, und diese, sammt den zugehörigen geradlinigen Röhrenflächen, hinterher irgend einer Bewegung unterwirft.

NEAPEL, 5. *November*, 1909.

On Steinerians of Quartic Surfaces.

BY JOHN N. VAN DER VRIES.

Connected with each algebraic surface in three-dimensional space are a number of covariant surfaces. Among these the Steinerian and the Hessian play an important rôle. The Hessian may be defined as the locus of points which are double points on first polars, or as the locus of points whose polar quadrics have double points. The Steinerian, the reciprocal surface to the Hessian, may be defined as the locus of points whose first polars have double points, or as the locus of points which are double points on polar quadrics, *i. e.*, which are the vertices of these quadric cones.* The Hessians and the Steinerians of surfaces of the third order have been investigated by Salmon, Cayley and others. The Hessian has also been investigated for surfaces of the fourth and higher orders. In two-dimensional geometry, the Hessian and the Steinerian of plane curves have been treated quite fully by Hesse, Steiner, Salmon, Cayley and others. The corresponding work in one-dimensional geometry has been done by Professor Newson.

It is the object of this paper to investigate the properties of the Steinerians of surfaces of the fourth order. These Steinerians are surfaces of order 32; they are therefore not simple surfaces. The work of determining them is, as will be seen later, very complicated, and results have been obtained only in special cases. These special cases are treated in detail, the remainder of the paper being devoted to considering special features of the Steinerians of quartic surfaces having certain singularities.

A. *The Developable Quartic.*

The polars of any point on an algebraic surface with respect to the surface itself touch the surface at the point itself. The tangents to the two branches of the curves in which the tangent plane to the surface at that point intersect the surface or any of its polars down to the polar quadric are respectively coincident. The polar quadric of any parabolic point on the surface with respect to the surface itself is a cone. The point itself is not the vertex of this cone, but the tangent plane to the surface at this point touches the cone along a generating line. Every point on a developable surface is a parabolic point. The polar quadric of every point of this surface will be the tangent cone at such a point; and, as two consecutive generators meet in a point on the cuspidal edge, these tangent cones will have as their vertices the points of the cuspidal edge. It has been shown* that the Hessian of a developable quartic is equal to the square of the quartic itself; *i. e.*, every point of the surface is a point of the Hessian. The points of the Steinerian, therefore, which correspond to the points of the Hessian lying on the developable quartic exclusive of the cuspidal edge, are points of the cuspidal edge. The cuspidal edge of the developable quartic is a unodal line. A unode is a double point whose tangent cone breaks up into two coincident planes, every point of these planes thus being a double point. The double point of the polar quadric which corresponds to a point of the Hessian on the cuspidal edge is therefore any point of the unodal plane at that point. The points of the Steinerian corresponding to points on the cuspidal edge of the Hessian are therefore the points of the osculating plane. As this plane varies from point to point along the cuspidal edge and during its revolution takes in every point of space, it is evident that the Steinerian of the developable quartic fills all space. *There is thus no definite Steinerian.*

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*Salmon: "Geometry of Three Dimensions," 4th edition, pp. 492, 493.

This result can also be shown analytically. The developable quartic is the envelope of tangent lines to a twisted cubic, the coordinates of any point of which are given by

$$x : y : z : s = t^3 : t^2 : t : 1;$$

that is,

$$\begin{Vmatrix} x & y & z & s \\ t^3 & t^2 & t & 1 \end{Vmatrix} = 0.$$

The equation of the quartic is obtained by eliminating t from

$$\begin{Vmatrix} x & y & z & s \\ t^3 & t^2 & t & 1 \\ 3t^2 & 2t & 1 & 0 \end{Vmatrix} = 0.$$

Eliminating t from

$$y - 2zt + st^2 = 0, \quad \text{and} \quad x - 2yt + zt^2 = 0,$$

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* Salmon: "Geometry of Three Dimensions," p. 489.

we obtain as the equation of the quartic

$$\begin{vmatrix} x & -2y & z & 0 \\ 0 & x & -2y & z \\ y & -2z & s & 0 \\ 0 & y & -2z & s \end{vmatrix} = 0;$$

i. e.,

$$U \equiv (xs - yz)^2 - 4\,(ys - z^2)\,(xz - y^2) = 0;$$

$$xs - yz = ys - z^2 = xz - y^2 = 0$$

being the equations of the twisted cubic. Let $(\xi, \eta, \zeta, \sigma)$ be the coordinates of a point on the Steinerian, and (x, y, z, s) those of a point on the Hessian. The equation of the first polar of $(\xi, \eta, \zeta, \sigma)$ with respect to the surface $U = 0$ is

$$\xi L + \eta M + \zeta N + \sigma P = 0, \tag{I}$$

where L, M, N, P are the first derivatives of U with respect to x, y, z, and s. If the first polar of $(\xi, \eta, \zeta, \sigma)$ with respect to $U = 0$ has a double point, this double point must satisfy the derivatives of (I) with respect to x, y, z and s; viz.:

$$\begin{aligned} &s(\xi s - \eta z - \zeta y + \sigma x) - 2z(\eta s - 2\zeta z + \sigma y) + \sigma(xs - yz) - 2\zeta(ys - z^2) = 0, \\ &-z(\xi s - \eta z - \zeta y + \sigma x) - 2s(\xi z - 2\eta y + \zeta x) + 4y(\eta s - 2\zeta z + \sigma y) \\ &\qquad\qquad - \zeta(xs - yz) + 4\eta(ys - z^2) - 2\sigma(xz - y^2) = 0, \\ &-y(\xi s - \eta z - \zeta y + \sigma x) + 4z(\xi z - 2\eta y + \zeta x) - 2x(\eta s - 2\zeta z + \sigma y) \\ &\qquad\qquad - \eta(xs - yz) - 2\xi(ys - z^2) - 4\zeta(xz - y^2) = 0, \\ &x(\xi s - \eta z - \zeta y + \sigma x) - 2y(\xi z - 2\eta y + \zeta x) + \xi(xs - yz) - 2\eta(xz - y^2) = 0. \end{aligned}$$

The equation of the Hessian is obtained by eliminating ξ, η, ζ, and σ from these equations, and the equation of the Steinerian by eliminating x, y, z, and s. For points of the Hessian on the cuspidal edge

$$xs - yz = ys - z^2 = xz - y^2 = 0,$$

and the equations for the determination of the corresponding points of the Steinerian reduce to

$$\begin{aligned} s(\xi s - \eta z - \zeta y + \sigma x) - 2z(\eta s - 2\zeta z + \sigma y) \qquad\qquad\qquad\qquad &= 0, \\ -z(\xi s - \eta z - \zeta y + \sigma x) + 4y(\eta s - 2\zeta z + \sigma y) - 2s(\xi z - 2\eta y + \zeta x) &= 0, \\ -y(\xi s - \eta z - \zeta y + \sigma x) - 2x(\eta s - 2\zeta z + \sigma y) + 4z(\xi z - 2\eta y + \zeta x) &= 0, \\ x(\xi s - \eta z - \zeta y + \sigma x) \qquad\qquad\qquad\qquad - 2y(\xi z - 2\eta y + \zeta x) &= 0. \end{aligned} \tag{II}$$

Considering these equations as linear equations in the linear functions

$$\xi s - \eta z - \zeta y + \sigma x, \quad \eta s - 2\zeta z + \sigma y, \quad \text{and} \quad \xi s - 2\zeta z + \sigma y,$$

it is evident that

$$\left\|\begin{matrix} s & -z & -y & x \\ 0 & -2s & 4z & -2y \\ 2z & 4y & -2x & 0 \end{matrix}\right\| = 0;$$

that is,

$$\begin{aligned} x(xs-yz)-2y(xz-y^2) \qquad\qquad &= 0, \\ z(xz-y^2)-x(ys-z^2) &= 0, \\ -s(xz-y^2)+y(ys-z^2) &= 0, \\ s(xs-yz) \qquad\qquad -2z(ys-z^2) &= 0. \end{aligned}$$

These equations are not independent but are equivalent to two independent equations, viz.:

$$xs-yz=xz-y^2=ys-z^2=0,$$

i. e.,

$$\frac{x}{y}=\frac{y}{z}=\frac{z}{s}.$$

Thus only two of equations (II) are independent. Considering the first and the fourth, substituting t^3, t^2, t and 1 for x, y, z and s, we have

$$\begin{aligned} (\xi-\eta t-\zeta t^2+\sigma t^3)-2t(\eta-2\zeta t+\sigma t^2) \quad &= 0, \\ t^3(\xi-\eta t-\zeta t^2+\sigma t^3)-2t^3(\xi t-2\eta t^2+\zeta t^3) &= 0. \end{aligned}$$

These are, however, identical and equal to

$$\xi-3\eta t-3\zeta t^2-\sigma t^3=0,$$

the equation of the osculating plane of the twisted cubic, Q. E. D.

B. *The Surface* $ax^2s^2+by^2z^2+xyzs=0$.

This surface has four unodes, $(0,0,0,1)$, $(0,0,1,0)$, $(0,1,0,0)$ and $(1,0,0,0)$, and four double lines, viz.: $x=y=0$, $x=z=0$, $y=s=0$ and $z=s=0$. The polar of the point (ξ,η,ζ,σ) with respect to the surface is

$$\xi(2axs^2+yzs)+\eta(2byz^2+xzs)+\zeta(2by^2z=xys)+\sigma(2ax^2s+xyz)=0.$$

If this polar has a double point, this point must satisfy the equations

$$\left.\begin{aligned} &\xi(2as^2) &&+\eta(zs) &&+\zeta(ys) &&+\sigma(4axs+yz) &&= 0, \\ &\xi(zs) &&+\eta(2bz^2) &&+\zeta(4byz+xs) &&+\sigma(xz) &&= 0, \\ &\xi(ys) &&+\eta(4byz+xs) &&+\zeta(2by^2) &&+\sigma(xy) &&= 0, \\ &\xi(4axs+yz) &&+\eta(xz) &&+\xi(xy) &&+\sigma(2ax^2) &&= 0. \end{aligned}\right\} \qquad \text{(III)}$$

The equation of the Hessian reduces to

$$[2byz+xs]\,[2axs+yz]\,[2ax^2s^2-xyzs(1-12ab)+2by^2z^2]=0.$$

It is thus composed of two quadric surfaces, having the double lines of the quartic as simple lines, and a quartic surface having the same singularities as the original quartic.

The equation of the Steinerian is obtained by eliminating x, y, z and s from equations (III) above; this is accomplished by multiplying the four equations successively by x, y, z and s, and then eliminating the variables from the sixteen equations thus obtained and the four derivatives with respect to x, y, z and s of the equation of the Jacobian. The equation of the Jacobian is

$$J \equiv \begin{vmatrix} 4a\sigma s, & \zeta s+\sigma z, & \eta s+\sigma y, & 4a\xi s+\eta z+\zeta y+4a\sigma x \\ \zeta s+\sigma z, & 4b\zeta z, & \zeta s+4b\eta z+4b\zeta y+\sigma x, & \xi z+\zeta x \\ \eta s+\sigma y, & \xi s+4b\eta z+4b\zeta y+\sigma x, & 4b\eta y, & \xi y+\eta x \\ 4a\xi s+\eta z+\zeta y+4a\sigma x, & \xi z+\zeta x, & \xi y+\eta x, & 4a\xi x \end{vmatrix} = 0;$$

that is,

$$\begin{aligned} J \equiv\ & a_1x^4 + b_1y^4 + c_1z^4 + d_1s^4 + a_2x^3y + b_2x^3z + c_2ys^3 + d_2xy^3 + e_2zs^3 + f_2y^3s + g_2xz^3 \\ & + h_2z^3s + i_2yz^3 + j_2xs^3 + k_2x^3s + l_2y^3z + a_3x^2y^2 + b_3x^2z^2 + c_3y^2s^2 + d_3z^2s^2 \\ & + e_3x^2s^2 + f_3y^2z^2 + a_4x^2yz + b_4xy^2s + c_4xz^2s + d_4yzs^2 + e_4x^2ys + f_4xy^2z \\ & + g_4yz^2s + h_4xzs^2 + i_4x^2zs + j_4y^2zs + k_4xyz^2 + l_4xys^2 + m_4xyzs = 0, \end{aligned}$$

where

$$\begin{aligned} a_1 &= 16a^2\sigma^4, \\ b_1 &= 16b^2\zeta^4, \\ c_1 &= 16b^2\eta^4, \\ d_1 &= 16a^2\xi^4, \\ a_2 &= 128a^2b\zeta\sigma^3, \\ b_2 &= 128a^2b\eta\sigma^3, \\ c_2 &= 128a^2b\xi^3\zeta, \\ d_2 &= 128ab^2\zeta^3\sigma, \\ e_2 &= 128a^2b\xi^3\eta, \\ f_2 &= 128ab^2\xi\zeta^3, \\ g_2 &= 128ab^2\eta^3\sigma, \\ h_2 &= 128ab^2\xi\eta^3, \\ i_2 &= 48b^2\eta^3\zeta - 8b\xi\eta^2\sigma, \\ j_2 &= 48a^2\xi^3\sigma - 8a\xi^2\eta\zeta, \\ k_2 &= 48a^2\xi\sigma^3 - 8a\eta\zeta\sigma^2, \\ l_2 &= 48b^2\eta\zeta^3 - 8b\xi\zeta^2\sigma, \\ a_3 &= 32ab\zeta^2\sigma^2 + 256a^2b^2\zeta^2\sigma^2, \\ b_3 &= 32ab\eta^2\sigma^2 + 256a^2b^2\eta^2\sigma^2, \\ c_3 &= 32ab\xi^2\zeta^2 + 256a^2b^2\xi^2\zeta^2, \end{aligned}$$

$$
\begin{aligned}
d_3 &= 32ab\xi^2\eta^2 + 256a^2b^2\xi^2\eta^2,\\
e_3 &= 64a^2\xi^2\sigma^2 - 16a\xi\eta\zeta\sigma,\\
f_3 &= 64b^2\eta^2\zeta^2 - 16b\xi\eta\zeta\sigma,\\
a_4 &= -4\eta\zeta\sigma^3 + 128ab\eta\zeta\sigma^2 + 256a^2b^2\eta\zeta\sigma^2 - 8a\xi\sigma^3,\\
b_4 &= -4\xi\zeta^2\sigma + 128ab\xi\zeta^2\sigma + 256a^2b^2\xi\zeta^2\sigma - 8b\eta\zeta^3,\\
c_4 &= -4\xi\eta^2\sigma + 128ab\xi\eta^2\sigma + 256a^2b^2\xi\eta^2\sigma - 8b\eta^3\zeta,\\
d_4 &= -4\xi^2\eta\zeta + 128ab\xi^2\eta\zeta + 256a^2b^2\xi^2\eta\zeta - 8a\xi^3\sigma,\\
e_4 &= -4\eta\zeta^2\sigma + 16a\xi\zeta\sigma^2 + 256a^2b\xi\zeta\sigma^2 - 16ab\eta\zeta^2\sigma,\\
f_4 &= -4\xi\zeta\sigma^2 + 16b\eta\zeta^2\sigma + 256ab^2\eta\zeta^2\sigma - 16ab\xi\zeta\sigma^2,\\
g_4 &= -4\xi^2\eta\sigma + 16b\xi\eta^2\zeta + 256ab^2\xi\eta^2\zeta - 16ab\xi^2\eta\sigma,\\
h_4 &= -4\xi\eta^2\zeta + 16a\xi^2\eta\sigma + 256a^2b\xi^2\eta\sigma - 16ab\xi\eta^2\zeta,\\
i_4 &= -4\eta^2\zeta\sigma + 16a\xi\eta\sigma^2 + 256a^2b\xi\eta\sigma^2 - 16ab\eta^2\zeta\sigma,\\
j_4 &= -4\xi^2\zeta\sigma + 16b\xi\eta\zeta^2 + 256ab^2\xi\eta\zeta^2 - 16ab\xi^2\zeta\sigma,\\
k_4 &= -4\xi\eta\sigma^2 + 16b\eta^2\zeta\sigma + 256ab^2\eta^2\zeta\sigma - 16ab\xi\eta\sigma^2,\\
l_4 &= -4\xi\eta\zeta^2 + 16a\xi^2\zeta\sigma + 256a^2b\xi^2\zeta\sigma - 16ab\xi\eta\zeta^2,\\
m_4 &= +256a^2b^2\xi\eta\zeta\sigma + 256ab\xi\eta\zeta\sigma - 16a\xi^2\sigma^2 - 16b\eta^2\zeta^2.
\end{aligned}
$$

The Steinerian is

$$
\left.\begin{array}{cccccccccccccccccccc}
a_1, & d_2, & g_2, & j_2, & 3a_2, & 3b_2, & 3k_2, & 2a_3, & f_4, & b_4, & 2b_3, & k_4, & c_4, & 2e_3, & l_4, & h_4, & 2a_4, & 2e_4, & 2i_4, & m_4\\
a_2, & 4b_1, & i_2, & c_2, & 2a_3, & a_4, & e_4, & 3d_2, & 3l_2, & 3f_2, & k_4, & 2f_3, & g_4, & l_4, & 2c_3, & d_4, & 2f_4, & 2b_4, & m_4, & 2j_4\\
b_2, & l_2, & 4c_1, & e_2, & a_4, & 2b_3, & i_4, & f_4, & 2f_3, & j_4, & 3g_2, & 3i_2, & 3h_2, & h_4, & d_4, & 2d_3, & 2k_4, & m_4, & 2c_4, & 2g^4\\
k_2, & f_2, & h_2, & 4d_1, & e_4, & i_4, & 2e_3, & b_4, & j_4, & 2c_3, & c_4, & g_4, & 2d_3, & 3j_2, & 3c_2, & 3e_2, & m_4, & 2l_4, & 2h_4, & 2d_4\\
0, & 0, & 0, & 0, & 0, & 0, & 4a\sigma, & 0, & 0, & 0, & 0, & 0, & 0, & 2a\xi, & 0, & 0, & \sigma, & \zeta, & \eta, & 0\\
0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \sigma, & \zeta, & 0, & 0, & 0, & 0, & 2a\xi, & 0, & 0, & 4a\sigma, & 0, & \eta\\
0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \sigma, & \eta, & 0, & 0, & 2a\xi, & 0, & 0, & 4a\sigma, & \zeta\\
0, & 0, & 0, & 2a\xi, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 4a\sigma, & \zeta, & \eta, & 0, & 0, & 0, & \sigma\\
0, & 0, & 0, & 0, & 0, & \sigma, & \zeta, & 0, & 0, & 0, & 2b\eta, & 0, & 0, & 0, & 0, & 0, & 4b\zeta, & 0, & \xi, & 0\\
0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 4b\zeta, & 0, & 0, & 2b\eta, & 0, & 0, & 0, & 0, & \sigma, & \zeta, & 0, & \xi\\
0, & 0, & 2b\eta, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \sigma, & 4b\zeta, & \xi, & 0, & 0, & 0, & 0, & 0, & \zeta, & 0\\
0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 2b\eta, & \zeta, & 0, & \xi, & 0, & 0, & \sigma, & 4b\zeta\\
0, & 0, & 0, & 0, & \sigma, & 0, & \eta, & 2b\zeta, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 4b\eta, & \xi, & 0, & 0\\
0, & 2b\zeta, & 0, & 0, & 0, & 0, & 0, & \sigma, & 4b\eta, & \xi, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \eta, & 0, & 0\\
0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 2b\zeta, & 0, & 0, & 4b\eta, & 0, & 0, & 0, & 0, & \sigma, & 0, & \eta, & \xi\\
0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 2b\zeta, & 0, & 0, & 0, & \eta, & \xi, & 0, & 0, & \sigma, & 0, & 4b\eta\\
2a\sigma, & 0, & 0, & 0, & \zeta, & \eta, & 4a\xi, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \xi, & 0, & 0, & 0\\
0, & 0, & 0, & 0, & 2a\sigma, & 0, & 0, & \zeta, & \xi, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \eta, & 4a\xi, & 0, & 0\\
0, & 0, & 0, & 0, & 0, & 2a\sigma, & 0, & 0, & 0, & 0, & \eta, & \xi, & 0, & 0, & 0, & 0, & \zeta, & 0, & 4a\xi, & 0\\
0, & 0, & 0, & 0, & 0, & 0, & 2a\sigma, & 0, & 0, & 0, & 0, & 0, & 0, & 4a\xi, & 0, & 0, & 0, & \zeta, & \eta, & \xi
\end{array}\right| = 0
$$

By continued reduction this reduces to

$$2304a^4b^4(1-4ab)^4\xi^4\eta^4\zeta^4\sigma^4\begin{vmatrix} A, & F, & H, & E \\ G, & B, & E, & J \\ I, & E, & C, & L \\ E, & K, & M, & D \end{vmatrix}=0,$$

where

$$\begin{aligned}
A &\equiv -32\eta\zeta\sigma^2+512ab\eta\zeta\sigma^2-1536a^2b^2\eta\zeta\sigma^2,\\
B &\equiv -32\xi\zeta^2\sigma+512ab\xi\zeta^2\sigma-1536a^2b^2\xi\zeta^2\sigma,\\
C &\equiv -32\xi\eta^2\sigma+512ab\xi\eta^2\sigma-1536a^2b^2\xi\eta^2\sigma,\\
D &\equiv -32\xi^2\eta\zeta+512ab\xi^2\eta\zeta-1536a^2b^2\xi^2\eta\zeta,\\
E &\equiv \quad 16\xi\eta\zeta\sigma+128ab\xi\eta\zeta\sigma-768a^2b^2\xi\eta\zeta\sigma,\\
F &\equiv -16\eta\zeta^2\sigma+64ab\eta\zeta^2\sigma+96a\xi\zeta\sigma^2-384a^2b\xi\zeta\sigma^2,\\
G &\equiv -16\xi\zeta\sigma^2-64ab\xi\zeta\sigma^2+96b\eta\zeta^2\sigma-384ab^2\eta\zeta^2\sigma,\\
H &\equiv -16\eta^2\zeta\sigma-64ab\eta^2\zeta\sigma+96a\xi\eta\sigma^2-384a^2b\xi\eta\sigma^2,\\
I &\equiv -16\xi\eta\sigma^2-64ab\xi\eta\sigma^2+96b\eta^2\zeta\sigma-384ab^2\eta^2\zeta\sigma,\\
J &\equiv -16\xi^2\zeta\sigma-64ab\xi^2\zeta\sigma+96b\xi\eta\zeta^2-384ab^2\xi\eta\zeta^2,\\
K &\equiv -16\xi\eta\zeta^2-64ab\xi\eta\zeta^2+96a\xi^2\zeta\sigma-384a^2b\xi^2\zeta\sigma,\\
L &\equiv -16\xi^2\eta\sigma-64ab\xi^2\eta\sigma+96b\xi\eta^2\zeta-384ab^2\xi\eta^2\zeta,\\
M &\equiv -16\xi\eta^2\zeta-64ab\xi\eta^2\zeta+96a\xi^2\eta\sigma-384a^2b\xi^2\eta\sigma.
\end{aligned}$$

This reduces to

$$\begin{aligned}
4076873488a^4b^4(1-4ab)^{10}\xi^7\eta^7\zeta^7\sigma^7\{8(a\xi^2\sigma^2+b\eta^2\zeta^2+\xi\eta\zeta\sigma)&\\
-9(1-4ab)(1+12ab)\xi\eta\zeta\sigma\}&=0;
\end{aligned}$$

that is, *each of the four unodal planes seven times and a quartic having all the singularities of the original quartic.*

C. *The Surface* $x^2y^2=(ax+by)^2(xz+ys)$.

This is a quartic with a triple line in which two of the sheets which meet in the triple line unite in a single cuspidal sheet. The four equations which must be satisfied by points in order that they may be double points on the first polars of $(\xi, \eta, \zeta, \sigma)$ are

$$\begin{aligned}
A\xi+C\eta+D\zeta+F\sigma&=0,\\
C\xi+B\eta+E\zeta+G\sigma&=0,\\
D\xi+E\eta\qquad\qquad&=0,\\
F\xi+G\eta\qquad\qquad&=0,
\end{aligned}$$

where

$$\begin{aligned}
A &\equiv 2y^2 - 2a^2(xz + ys) - 4az(ax + by),\\
B &\equiv 2x^2 - 2b^2(xz + ys) - 4bs(ax + by),\\
C &\equiv 4xy - 2(bz + as)(ax + by) - 2ab(xz + ys),\\
D &\equiv -(ax + by)^2 - 2ax(ax + by),\\
E &\equiv -2bx(ax + by),\\
F &\equiv -2ay(ax + by),\\
G &\equiv -(ax + by)^2 - 2by(ax + by).
\end{aligned}$$

Eliminating ξ, η, ζ and σ, the equation of the Hessian is found to be

$$\begin{vmatrix} A, & C, & D, & F \\ C, & D, & E, & G \\ D, & E, & 0, & 0 \\ F, & G, & 0, & 0 \end{vmatrix} = 0,$$

which reduce to

$$(ax + by)^8 = 0.$$

Thus for every point of the Hessian $ax + by = 0$, and the corresponding points of the Steinerian satisfy the equations

$$\xi\,[2y^2 - 2a^2(xz + ys)] + \eta\,[4xy - 2ab(xz + ys)] = 0$$

and

$$\xi\,[4xy - 2ab(xz + ys)] + \eta\,[2x^2 - 2b^2(xz + ys)] = 0.$$

These are the equations of the line $\xi = \eta = 0$; *i. e.*, the triple line $x = y = 0$. But, for any point of the Hessian on the triple line $x = y = 0$, the equations for determining the corresponding points of the Steinerian vanish identically; that is, *the Steinerian is composed of all the points in space.*

It can readily be seen by similar reasoning that *the Steinerian of every quartic having a triple line is satisfied by all points in space.*

D. *The Quartic* $u_4 + zu_3 + sv_3 + z^2t_2 + zsu_2 + s^2v_2 = 0$.

In this equation u_4, u_3, etc., are functions of x and y of degrees equal to their respective subscripts, the line xy thus being a nodal line on the quartic. The surface, the Hessian and the Steinerian intersect in this line xy. If the last three terms form a perfect square $(xz + ys)^2$, the line is cuspidal; for every point of the Hessian on this line, the corresponding points of the Steinerian satisfy the equation $z\xi + s\eta = 0$. As the point moves along this scrolar line, the plane $z\xi + s\eta$ revolves about the line and *the Steinerian is thus satisfied by all points in space.*

In case the last three terms reduce to the forms x^2zs or x^2z^2, the cuspidal tangent plane remains the same for all points on it, and the points of the Steinerian corresponding to points on this unodal line are points of this unodal plane. But in these two cases there are either one or two triple points on the line, and *in every such case the points of the Steinerian are all the points in space.*

E. *The Quartic Scroll* $(xz-y^2)^2 - mys(xz-y^2) + s^2(axy+by^3) = 0.$

This scroll has the line $x=y=0$ as a nodal line and $xz-y^2=s=0$ as a nodal conic. The Hessian reduces to

$$\begin{aligned}4(xz-y^2)^3\{m^2(xz-3y^2)+8(axy+by^2)\}+4ms(xz-y^2)^2\{4ax(xz+y^2)\\+4y^2(axy+by^2)-3m^2y^3+8bxyz\}+s^2(xz-y^2)\{16x(ay+3bz)(axy+by^2)\\-m^2y(4ax-m^2y)(xz+y^2)+6bm^2y^2(2xs-y^2)-2(m^4+8b^2)y^4+48a^2x^3z\}\\+2mys^3\{ax(4ax-m^2y)(xz+y^2)-2by[6ax^2z+8bxyz+2axy^2\\-(m^2+4b)y^3]\}-ys^4\{8ax^3-a^2(m^2-4b)x^2y+4b^2m^2y^3\}=0,\end{aligned}$$

and has the nodal conic as a triple conic and the double line as a quadruple line. For a point of the Hessian on the line $x=y=0$, the corresponding points of the Steinerian satisfy

$$\xi(2z^2)+\eta(mzs+as^2)=0$$

and

$$\xi(mzs+as^2)+\eta(2bs^2)=0,$$

which is the line $\xi=\eta=0$ (that is, the nodal line), *the original surface, the Hessian and the Steinerian thus intersecting in this line.*

Two points of the Hessian are pinch-points on the quartic; viz.,

$$x:y:z:s=0:0:-a:m-2\sqrt{b}$$

and

$$x:y:z:s=0:0:-a:m+2\sqrt{b}.$$

Substituting the coordinates of the first in the equations for determining the Steinerian, the points of the Steinerian corresponding to this point are found to satisfy the equations

$$2a^2\xi+(-2am\sqrt{b}+4ab)\eta=0$$

and

$$(-2am\sqrt{b}+4ab)\xi+2b(m-2\sqrt{b})\eta=0,$$

which reduce to the same equation; viz.,

$$a\xi-\sqrt{b}(m-2\sqrt{b})\eta=0,$$

the pinch-plane at the point. Similarly, for the other pinch-point.

For points of the Hessian on the nodal conic $xz-y^2=s=0$, the equations for determining the points of the Hessian and the Steinerian reduce to

$$2(z\xi - 2y\eta + x\zeta) + my\sigma = 0,$$
$$m(z\xi - 2y\eta + x\zeta) + 2(ax + by)\sigma = 0,$$

which are satisfied by

$$z\xi - 2y\eta + x\zeta = \sigma = 0;$$

that is, the tangent line to the conic at any point of it. As the point passes around the conic, this line sweeps over the plane. *The plane of the nodal conic is thus part of the Steinerian.*

The points $x : y : z : s = 0 : 0 : 1 : 0$

and $x : y : z : s = (m^2 - 4b)^2 : 4a(m^2 - 4b) : 16a^2 : 0$

are pinch-points on this conic. By similar reasoning to the above it is seen that

$$\xi = 0 \quad \text{and} \quad 16a^2\xi - 8a(m^2 - 4b)\eta + (m^2 - 4b)^2\zeta + 2am(m^2 - 4b)\sigma = 0,$$

the pinch-planes at the two points, are also parts of the Steinerian.

Similarly, in the case of the quartic with a nodal conic having for its equation

$$\phi^2 - 4P^2\psi = 0,$$

where

$$\phi \equiv ax^2 + by^2 + cz^2 + ds^2,$$
$$\psi \equiv a_1x^2 + b_1y^2 + c_1z^2 + d_1s^2,$$
$$P \equiv z,$$

for points of the Hessian on the nodal conic $ax^2 + by^2 + da^2 = z = 0$, the corresponding points of the Steinerian satisfy

$$ax\xi + by\eta + ds\sigma = 0, \quad \text{and} \quad \zeta = 0;$$

that is, *lie on the plane of the nodal conic* $z = 0$.

F.

In the case of the ruled quartic

$$x^2(az^2 + 2hzs + bs^2) + 2xy(a'z^2 + 2h'zs + b's^2) + y^2(a''z^2 + 2h''zs + b''s^2) = 0,$$

having the two non-intersecting double lines xy and zs, for points of the Hessian on either nodal line the corresponding points of the Steinerian are on the same double line; that is, *the surface, the Steinerian and the Hessian intersect in the double line.*

The same is true for the quartic

$$x^2y^2 + 2mxyzs + s^2u_2 = 0$$

(where u_2 is a quadratic function of x, y, z and s) having xs and ys as intersecting double lines; and it is also true in the case of every double line on the quartic.

LAWRENCE, KANSAS, *November*, 1908.

On the Determination of the Ternary Modular Groups.

BY R. L. BÖRGER.

Introduction.

The subgroups of the linear fractional group $z' = \frac{\alpha z + \beta}{\gamma z + \delta}$ in $GF[p^n]$ have been determined.[*] Closely related to this is the determination of the binary groups of determinant unity in the $GF[p^n]$.[†] The subgroups of the special ternary group $GF[p]$ have been found.[‡] The object of the present paper is to apply the method of the paper last cited to the group in $GF[p^2]$. The order of the special ternary linear group in $GF[p^2]$ is $p^6(p^6-1)(p^4-1)$. The subgroups of order a multiple of p, $p^i M (i = 6, 5, 4)$ (M prime to p) are determined in this paper. The determination of the groups of these high orders applies in particular to the question of the minimum index of a subgroup of total group. For $i = 3, 2, 1$ the method can not be employed and recourse must be had to special devices.

Subgroups of Order a Power of p.

1. The order of the special linear ternary group G in the $GF[p^2]$ is $p^6(p^6-1)(p^4-1)$. Within G every subgroup of order a power of p is conjugate with a subgroup of the group G_{p^6} of the operations

$$[\alpha] = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_{21} & 1 & 0 \\ \alpha_{31} & \alpha_{32} & 1 \end{pmatrix} \qquad (1)$$

with the α_{ij} arbitrary in the $GF[p^2]$.

Since[§] the commutator of G_{p^6} is of order p^2 and every operation is of order p, $(p \neq 2)$, there are $\dfrac{p^4-1}{p-1}$ subgroups of order p^5. They are given by a linear relation modulo p,

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[*] Gierster, Wiman; Moore, *University of Chicago Decennial Publications*, Vol. IX (1903), pp. 141–190; Dickson: "Linear Groups."

[†] Made by Dickson — unpublished.

[‡] Dickson, AMERICAN JOURNAL, Vol. XXVII, pp. 189–202.

[§] The results in §§ 1–3 are proved in detail in AMERICAN JOURNAL, Vol. XXVII, pp. 289–293.

$$l\,(a_{210},\ a_{211},\ a_{320},\ a_{321}) \equiv 0. \tag{2}$$

2. The commutator subgroup K of any group L defined by a relation (2) is of order p^3.

THEOREM: Any group L_{p^5} defined by one linear relation (2) has exactly p^3+p+1 subgroups of order p^4, defined by two independent relations (2). The number of distinct subgroups of order p^4 of G_{p^6} is thus

$$p^4+p^3+2p^2+p+1.$$

They are as follows:

(I$_a$) $a_{210} = g\,a_{320} + h\,a_{321}$, $a_{211} = l\,a_{320} + m\,a_{321}$ $[\alpha_{21} = t\,\alpha_{32}]$.
(I) $a_{210} = g\,a_{320} + h\,a_{321}$, $a_{211} = l\,a_{320} + m\,a_{321}$ $[\alpha_{21} \neq t\,\alpha_{32}]$.
(II) $a_{321} = h\,a_{320}$, $a_{211} = l\,a_{210} + m\,a_{320}$.
(III) $a_{331} = h\,a_{320}$, $a_{210} = m\,a_{320}$.
(IV) $a_{320} \equiv 0$, $a_{211} = l\,a_{211} + m\,a_{321}$.
(V) $a_{320} \equiv 0$, $a_{210} = m\,a_{321}$.
(VI) $a_{320} \equiv 0$, $a_{321} \equiv 0$.

THEOREM. The only commutative subgroups of order p^4 are those defined by $\{\alpha_{32} = 0\}$ and $\{\alpha_{21} = t\,\alpha_{32}\}$. For the remaining subgroups of order p^4 the commutator subgroup is of order p.

3. THEOREM. For $p > 2$ the subgroups G_{p^3} of the G_{p^4}, given in the first column, are obtained by annexing to the two relations defining G_{p^4} an independent linear homogeneous relation between the elements given in the second column.

G_{p^4}.	ELEMENTS.		
(I$_a$)	a_{320},	a_{321},	$a_{310} - \frac{1}{2}\,g\,a^2_{320} - h\,a_{320}\,a_{321} - \frac{1}{2}(\nu g + \mu h)\,a^2_{320}$, $a_{311} - \dfrac{h}{2\nu}a^2_{321} - \left(g + \mu\dfrac{h}{\nu}\right)a_{320}\,a_{321} - \frac{1}{2}\left(h + \dfrac{\mu^2 h}{\nu} + \mu g\right)a^2_{321}$.
(I)	a_{320},	a_{321},	$(m - g - l\mu)\,a_{310} - (h - \nu l)\,a_{311} - u\,a^2_{320} - v\,a_{320}\,a_{321} - w\,a^2_{321}$.
(II)	a_{320},	a_{210},	$(l + h + lh\mu)\,a_{310} - (1 + hl\nu)\,a_{311} + \frac{1}{2}\,m\,(1 + h\mu - h^2\nu)\,a^2_{320}$.
(III)	a_{320},	a_{211},	$(1 + h\mu)\,a_{310} - h\nu\,a_{311} - \dfrac{m}{2}(1 + h\mu - h^2\nu)\,a^2_{320}$.
(IV)	a_{321},	a_{210},	$(1 + l\mu)\,a_{310} - l\nu\,a_{311} - \frac{1}{2}\,m\nu\,a^2_{321}$.
(V)	a_{321},	a_{211},	$\mu\,a_{310} - \nu\,a_{311} + \frac{1}{2}\,m\nu\,a^2_{321}$.
(VI)	a_{210},	a_{211},	a_{310}, a_{311}.

The u, v and w used in (I) denote

$$u \equiv \tfrac{1}{2} g (m - g - l\mu) - \tfrac{1}{2} l (h - \nu l),$$
$$v \equiv m \nu l - h g - h l \mu,$$
$$w \equiv \tfrac{1}{2} m \nu (m - g - l\mu) - \tfrac{1}{2} (h - l\nu)(h + m\mu).$$

The Galois Field is defined by the irreducible congruence $\rho^2 \equiv \mu\rho + \nu \pmod{p}$.

4. The subgroups of order p^3 of any abelian group G_{p^4} represented in the first column are given by annexing, to the relations defining the G_{p^4}, an independent linear homogeneous relation between the elements of the second column.

G_{p^4}	ELEMENTS.
(I_a)	a_{320}, a_{321}, $a_{310} - \tfrac{1}{2} g a_{320}^2 - h a_{320} a_{321} - \tfrac{1}{2}(\nu g + \mu h) a_{321}^2$, $a_{311} - \dfrac{h}{2\nu} a_{321}^2 - \left(g + \dfrac{\mu h}{\nu}\right) a_{320} a_{321} - \tfrac{1}{2}\left(h + \dfrac{\mu^2 h}{\nu} + \mu g\right) a_{321}^2$.
$\mathrm{I}_{1,3}$	a_{320}, $a_{310} - \tfrac{1}{2} g a_{320}^2$, $a_{311} - \tfrac{1}{2} l a_{320}^2$.
$\mathrm{I}_{2,3}$	a_{321}, $a_{310} - \dfrac{\nu h}{2} a_{321}^2$, $a_{311} - \tfrac{1}{2}(h + \mu m) a_{321}^2$.
I_3	a_{320}, $a_{310} - \tfrac{1}{2}\lambda a_{320}^2$, $a_{311} - \tfrac{1}{2}\eta a_{320}^2$.
$\mathrm{II}_{1,3}$	a_{320}, $a_{310} - \tfrac{1}{2}\nu h m a_{320}^2$, $a_{311} - \tfrac{1}{2}(m + \mu h m) a_{320}^2$.
$\mathrm{II}_{2,3}$	a_{210}, a_{310}, a_{311}.
II_3	a_{320}, $a_{310} - \tfrac{1}{2}[k + \nu h(lk + m)] a_{320}^2$, $a_{311} - \tfrac{1}{2}\{lk + m + hk + \mu h(lk + m)\} a_{320}^2$.
$\mathrm{III}_{1,3}$	a_{320}, $a_{310} - \tfrac{1}{2} m a_{320}^2$, $a_{311} - \dfrac{hm}{2} a_{320}^2$.
$\mathrm{III}_{2,3}$	a_{211}, a_{310}, a_{311}.
III_3	a_{320}, $a_{311} - \tfrac{1}{2}(m + \nu h k) a_{320}^2$, $a_{311} - \tfrac{1}{2}(k + hm + \mu h k) a_{320}^2$.
$\mathrm{IV}_{1,3}$	a_{321}, $a_{310} - \tfrac{1}{2} m \nu a_{321}^2$, $a_{311} - \tfrac{1}{2} m \nu a_{321}^2$.
$\mathrm{IV}_{2,3}$	a_{210}, a_{310}, a_{311}.
IV_3	a_{321}, $a_{310} - \dfrac{\nu}{2}(kl + m) a_{321}^2$, $a_{311} - \dfrac{m\nu}{2} a_{321}^2$.
$\mathrm{V}_{1,3}$	a_{331}, a_{310}, $a_{311} - \tfrac{1}{2} m a_{331}^2$.
$\mathrm{V}_{2,3}$	a_{211}, a_{310}, a_{311}.
V_3	a_{321}, $a_{310} - \nu \dfrac{k}{2} a_{321}^2$, $a_{311} - \tfrac{1}{2}(m + mk) a_{321}^2$.
VI	a_{210}; a_{211}, a_{310}, a_{311}.

$$[\lambda \equiv g + hk + \nu k l + \nu k^2 m, \quad \eta \equiv l + mk + gk + hk^2 + \mu k l + \mu k^2 m,$$
$$k = -l_1/l_2.]$$

The notation in first column meaning that coefficients corresponding to the subscripts are 0, the others being destinct from 0. Thus: $\mathrm{V}_{1,3}$ means that linear relation defining the G_{p^3} has $l_1 = l_3 = 0$, $l_2 \neq 0$; V_3 means that linear relation defining the G_{p^3} has $l_3 = 0$, $l_1 \neq 0$, $l_2 \neq 0$.

The remaining subgroups of order p^3 are non-abelian.

*Lemma.** The commutator subgroup of any non-abelian group of order p^3 is of order p.

Proof: If G_p is a subgroup of invariant operations, the quotient group G_{p^3}/G_p is of order p^2. Hence G_p contains the commutator subgroup of G_{p^3}. Since G_{p^3} is non-abelian the commutator is G_p.

It follows that the number of subgroups of order p^2 of G_{p^3} (G_{p^3} being non-abelian and operations all of order p) is $p+1$. They are given by annexing to the relations defining G_{p^3} an independent linear homogeneous relation between the elements of the first two columns in § 4.

The subgroups of order p are easily found. They will not be written here.

Subgroups of Order a Multiple of p.

5. *Lemma* I. Every operation (β_{ij}) permutable with a subgroup, not the identity, of G_{p^3} is contained in the set:

$$\begin{pmatrix} \beta_{11} & 0 & 0 \\ \beta_{21} & \beta_{22} & 0 \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}; \quad \begin{pmatrix} \beta_{11} & \beta_{12} & 0 \\ \beta_{21} & \beta_{22} & 0 \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}; \quad \begin{pmatrix} \beta_{11} & 0 & 0 \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}. \qquad (3)$$

Proof: Among the conditions for $(\beta_{ij})[\alpha] = [\gamma](\beta_{ij})$ are:

$$\beta_{11} + \beta_{12}\alpha_{21} + \beta_{13}\alpha_{31} = \beta_{11}, \qquad (1)$$

$$\beta_{21} + \beta_{22}\alpha_{21} + \beta_{23}\alpha_{31} = \beta_{11}\gamma_{21} + \beta_{21}, \qquad (2)$$

$$\beta_{12} + \beta_{13}\alpha_{32} = \beta_{12}, \qquad (3)$$

$$\beta_{22} + \beta_{23}\alpha_{32} = \beta_{12}\gamma_{21} + \beta_{22}, \qquad (4)$$

$$\beta_{32} + \beta_{33}\alpha_{32} = \beta_{12}\gamma_{31} + \beta_{22}\gamma_{32} + \beta_{32}, \qquad (5)$$

$$\beta_{23} = \beta_{23} + \gamma_{21}\beta_{13}. \qquad (6)$$

If $\alpha_{21} \equiv 0$, $\alpha_{32} \not\equiv 0$, $\alpha_{31} \not\equiv 0$, then $\beta_{13} = \beta_{23} = 0$ by (1) and (2).

In like manner we treat the remaining cases and prove the lemma.

Lemma II. The only factors $\equiv 1 \pmod{p^4}$ of $(p^6 - 1)(p^4 - 1) \equiv \omega$ are 1 and ω. The proof is similar to that in the paper† referred to and will not be given here.

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* Miller, *Bulletin American Mathematical Society*, 1897–98, p. 137.

† Dickson, AMERICAN JOURNAL, Vol. XXVII, p. 191.

Lemma III. Any binary transformation $B = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$, $\gamma \neq 0$, and all the $E_\lambda = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$ generate every binary transformation T of determinant unity.

For proof, see reference above.

Lemma IV. If a group G_1, of order $p^\beta N$ (N prime to p), contains a subgroup of G_{p^α} $(\alpha > \beta)$ of order p^β and if P is an operation of G_1 permutable with G_{p^α}, then P is permutable with G_{p^β}.

Proof: Suppose P were not permutable with G_{p^β}. It would transform G_{p^β} into G'_{p^β}; and $\{G_{p^\beta}, G'_{p^\beta}\}$, being of order a power of p, would be of order $p^{2\beta-c}$ (where p^c is the order of the greatest subgroup common to G_{p^β} and G'_{p^β}). But for all values of $c < \beta$ we have $p^{2\beta-c} > p^\beta$. Hence $c = \beta$ and $G'_{p^\beta} \equiv G_{p^\beta}$.

Lemma V. The operations permutable with every group G_{p^3} defined by

$$a_{21} = 0, \quad k\,a_{331} + h\,a_{320} = 0$$

are contained in the set (3_1).

Proof: By Lemma I, they are contained in (3_2). Transforming G_{p^3} by (β_{ij}) $(\beta_{13} = \beta_{23} = 0)$ it becomes $G'_{p^3}[\gamma_{ij}]$, having

$$\gamma_{31} \equiv \frac{\beta_{11}}{\beta_{33}}\alpha_{31} + \frac{\beta_{21}}{\beta_{33}}\alpha_{32}, \qquad \gamma_{32} \equiv \frac{\beta_{12}}{\beta_{33}}\alpha_{31} + \frac{\beta_{22}}{\beta_{33}}\alpha_{32}.$$

Call

$$\frac{\beta_{11}}{\beta_{33}} \equiv c_{11} + \rho\,d_{11}, \quad \frac{\beta_{22}}{\beta_{33}} \equiv c_{22} + \rho\,d_{22},$$

$$\frac{\beta_{12}}{\beta_{33}} \equiv c_{12} + \rho\,d_{12}, \quad \frac{\beta_{21}}{\beta_{33}} \equiv c_{21} + \rho\,d_{21}.$$

Then $\gamma_{31} \equiv (c_{12}\,a_{310} + c_{22}\,a_{320} + \nu\,d_{12}\,a_{311} + \nu\,d_{22}\,a_{321})$

$$+ \rho\,(d_{12}\,a_{310} + c_{12}\,a_{311} + \mu\,d_{12}\,a_{311} + \mu\,d_{22}\,a_{321} + c_{22}\,a_{321} + d_{22}\,a_{320}).$$

Setting $\alpha_{32} = 0$ and recalling the definition of G_{p^3}, we get the relation:

$$(c_{12}\,h + d_{12}\,k)\,a_{310} + (\nu\,d_{12}\,h + c_{12}\,k + \mu\,d_{12}\,k)\,a_{311} = 0.$$

Since a_{310} and a_{311} are arbitrary, we have

$$c_{12}\,h + d_{12}\,k = 0, \quad c_{12}\,k + d_{12}\,(\nu\,h + \mu\,k) = 0.$$

Thus $c_{12} = d_{12} = 0$, since the determinant of their coefficients equals $\nu\,h^2 + \mu\,h\,k - k^2$, which vanishes only for the excluded case $h = k = 0$, since $\rho^2 \equiv \mu\,\rho + \nu$ was assumed irreducible.

6. Let H be a subgroup of order $p^6 N$ (N prime to p) normalized to contain G_{p^6}. Then, by methods similar to those employed in the paper referred to, we derive the following

THEOREM. *Within G every subgroup H of order a multiple of p^6 is conjugate with one of the following:*

a) *The group of all the $p^6 f.g$ operations (3_1) with $\alpha_{11}^f = 1$, $\alpha_{22}^g = 1$, $\alpha_{33} = \alpha_{11}^{-1}\alpha_{22}^{-1}$.*

b) *The group of all the $f p^6 (p^4 - 1)$ operations (3_2) with $\alpha_{33}^f = 1$, $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = \alpha_{33}^{-1}$.*

c) *The group of all the $f p^6 (p^4 - 1)$ operations (3_3) with $\alpha_{11}^f = 1$, $\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32} = \alpha_{11}^{-1}$, where f and g are any divisors of $p^2 - 1$.*

7. Let H be a subgroup of G of order $p^5 N$ (N prime to p) normalized to contain a subgroup G_{p^5} of G_{p^6}. Let the G_{p^5} be defined by

$$l_1 a_{210} + l_2 a_{211} + l_3 a_{320} + l_4 a_{321} = 0.$$

(I). We exclude for the present the special case $l_1 = l_2 = 0$ and the special case $l_3 = l_4 = 0$ (treated under II and III below), and treat here the general case. The number of conjugates to G_{p^5} is $1 + kp$, $1 + kp^2$, $1 + kp^3$, or $1 + kp^4$ (k being prime to p). We treat these cases in turn.

(I$_1$). Let the number of conjugates to G_{p^5} be $1 + kp$. Then two groups of order p^5 must contain a common subgroup of order p^4* and, by a theorem† due to Frobenius and Burnside independently, H must contain an operation P permutable with G_{p^4} but not with G_{p^5}. Since neither $l_1 = 0$, $l_2 = 0$ nor $l_3 = 0$, $l_4 = 0$ in the case before us, this common G_{p^4} must have neither $\alpha_{21} \equiv 0$ nor $\alpha_{32} \equiv 0$. Therefore the operations permutable with G_{p^4} are contained in (3_1) and P is also permutable with G_{p^6}. Hence P must, by Lemma IV, be permutable with G_{p^5}, in contradiction with the Frobenius-Burnside Theorem. Hence the number of conjugates to G_{p^5} can not be $1 + kp$.

(I$_2$). Suppose the number of conjugates to G_{p^5} is $1 + kp^2$. The case in which $\alpha_{31} \not\equiv 0$ in the common G_{p^3} has been covered by the previous argument.

Let next $\alpha_{31} \equiv 0$ in the common G_{p^3}. The groups of order p^3 are then defined by

$$\left.\begin{array}{l} a_{210} = a_{211} = 0 \\ a_{321} = h\, a_{320} \end{array}\right\}, \quad \left.\begin{array}{l} a_{210} = a_{211} = 0 \\ a_{320} = 0 \end{array}\right\}. \tag{5}$$

But for either of these groups, the operation P is contained in (3_1), and as before we conclude that the number of conjugates can not be $1 + kp^2$.

(I$_3$). Let the number of conjugates be $1 + kp^3$. For the common G_{p^4}, we have the following possibilities:

a) $\alpha_{21} \not\equiv 0$, $\alpha_{32} \not\equiv 0$; b) $\alpha_{21} \equiv 0$, $\alpha_{32} \not\equiv 0$; c) $\alpha_{21} \equiv 0$, $\alpha_{32} \equiv 0$; d) $\alpha_{21} \not\equiv 0$, $\alpha_{32} \equiv 0$.

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* Burnside: "Theory of Groups," p. 94, Cor. II. † *Op. cit.*, p. 97.

The operation P for the G_{p^3} defined in either a) or c) is contained in (3_1) and is therefore permutable with G_{p^6}. In b) we note that G_{p^6} must contain a subgroup G_{p^3} defined by (5_1) or (5_2). The operation permutable with the G_{p^3} in b) is contained either in (3_1) or (3_2), and we need only consider the case in which P is in (3_2). Since under this hypothesis P is not permutable with either of the groups G_{p^3} defined in (5_1), (5_2), it will transform either of them into a new G'_{p^3} and $\{G_{p^3}\, G'_{p^3}\}$ is of order p^4 with $\alpha_{21} \equiv 0$. This G_{p^4} is not a subgroup of G_{p^6}, but is permutable with it. Hence $\{G_{p^5},\ G_4\}$ must be of order p^6, in contradiction with the hypothesis on the order of H. Hence P must be contained in (3_1). It will then be permutable with G_{p^6}, contradicting the Frobenius-Burnside Theorem. A similar argument applies to the case d). We now conclude that the number of conjugates to G_{p^6} can not be $1 + kp^3$.

(I_4). By Lemma II the only divisor of the order of G of the form $1 + kp^4$ is $(p^6-1)(p^4-1)$. If the number of conjugates to G_{p^6} were $1 + kp^4$, the index of H under G would be p, whereas the order of $LF\{3, p^2\}$ is not a divisor of $p!$.

(II). Let H contain a G_{p^5} defined by $l_1 a_{210} + l_2 a_{211} = 0$. We shall prove that such a group H can not contain any operation (β_{ij}) not contained in (3_2). We first show that if any operation (β_{ij}) has β_{13}, β_{23} not both 0, then H contains an operation (γ_{ij}) having $\gamma_{12} = 0$, $\gamma_{13} = 0$, $\gamma_{23} \neq 0$.

a) If $\beta_{12} = \beta_{13} = 0$, while $\beta_{23} \neq 0$, this (β_{ij}) is such an operation.

b) The case $\beta_{12} \neq 0$, $\beta_{13} = 0$, may be disposed of by replacing (β_{ij}) by its inverse. We need then consider only

c) $\beta_{13} \neq 0$. Transforming G_{p^5} by β_{ij} we get (γ_{ij}), where

$$\gamma_{11} \equiv \overline{\beta}_{12}\beta_{11}\alpha'_{21} + \overline{\beta}_{13}\beta_{11}\alpha_{31} + \overline{\beta}_{13}\beta_{21}\alpha_{32} + 1, \tag{1}$$

$$\gamma_{12} \equiv \overline{\beta}_{12}\beta_{12}\alpha_{21} + \overline{\beta}_{13}\beta_{12}\alpha_{31} + \overline{\beta}_{13}\beta_{22}\alpha_{32}, \tag{2}$$

$$\gamma_{13} \equiv \overline{\beta}_{12}\beta_{13}\alpha_{21} + \overline{\beta}_{13}\beta_{13}\alpha_{31} + \overline{\beta}_{13}\beta_{23}\alpha_{32}, \tag{3}$$

$$\gamma_{21} \equiv \overline{\beta}_{23}\beta_{11}\alpha_{21} + \overline{\beta}_{23}\beta_{11}\alpha_{31} + \overline{\beta}_{23}\beta_{21}\alpha_{32}, \tag{4}$$

$$\gamma_{22} \equiv \overline{\beta}_{23}\beta_{12}\alpha_{21} + \overline{\beta}_{23}\beta_{12}\alpha_{31} + \overline{\beta}_{23}\beta_{22}\alpha_{33} + 1, \tag{5}$$

$$\gamma_{23} \equiv \overline{\beta}_{22}\beta_{13}\alpha_{21} + \overline{\beta}_{23}\beta_{13}\alpha_{31} + \overline{\beta}_{23}\beta_{23}\alpha_{32}, \tag{6}$$

$$\gamma_{31} \equiv \overline{\beta}_{32}\beta_{11}\alpha_{21} + \overline{\beta}_{33}\beta_{11}\alpha_{31} + \overline{\beta}_{33}\beta_{21}\alpha_{32}, \tag{7}$$

$$\gamma_{32} \equiv \overline{\beta}_{33}\beta_{12}\alpha_{21} + \overline{\beta}_{33}\beta_{12}\alpha_{31} + \overline{\beta}_{33}\beta_{22}\alpha_{32}, \tag{8}$$

$$\gamma_{33} \equiv \overline{\beta}_{32}\beta_{13}\alpha_{21} + \overline{\beta}_{33}\beta_{13}\alpha_{31} + \overline{\beta}_{33}\beta_{23}\alpha_{32} + 1, \tag{9}$$

where $\overline{\beta}_{ij}$ is the adjoint minor of β_{ji} in $|\beta_{ji}|$.

We wish to show that we can make $\gamma_{12}=\gamma_{13}=0$, $\gamma_{23}\neq 0$ by properly choosing α_{21}, α_{31}, α_{32}. We can by a normalization of the operator (β_{ij}) make $\overline{\beta}_{12}\neq 0$ without disturbing β_{13}; viz., by multiplying (β_{ij}) on the left by $B_{32\lambda}$. Now suppose that $\gamma_{23}=0$. Then, setting $\alpha_{32}=0$, we have the following equations:

$$\overline{\beta}_{22}\alpha_{21}+\overline{\beta}_{23}\alpha_{31}=0, \tag{6'}$$

$$\overline{\beta}_{32}\alpha_{21}+\overline{\beta}_{33}\alpha_{31}=0, \tag{9'}$$

$$\overline{\beta}_{12}\alpha_{21}+\beta_{13}\alpha_{31}=0. \tag{3'}$$

Multiplying (6′) by β_{32}, (9′) by β_{33} and (3′) by β_{31}, adding and applying $\Delta=1$, we get $\alpha_{31}=0$. This is impossible if $\overline{\beta}_{12}\neq 0$. It follows, then, that the operation (γ_{ij}) fulfils the required conditions if $\overline{\beta}_{13}\neq 0$. If $\overline{\beta}_{13}=0$ we can set $\alpha_{32}=0$ as before, and now put $\alpha_{21}=0$, thus making $\gamma_{12}=\gamma_{13}=0$. Then, if $\gamma_{23}=0$, we must have equations (6′), (9′), (3′) true for arbitrary values of α_{31}. From the previous argument these equations can only be simultaneously true when $\alpha_{31}=0$. Therefore the operation (γ_{ij}) always has $\gamma_{12}=\gamma_{13}=0$, $\gamma_{23}\neq 0$.

Transforming $[\alpha]$ by (γ_{ij}) we get an operation

$$\begin{pmatrix} 1 & 0 & 0\\ \delta_{21} & \delta_{22} & \delta_{23}\\ \delta_{31} & \delta_{32} & \delta_{33}\end{pmatrix}, \quad \begin{aligned}\delta_{21}&\equiv\overline{\gamma}_{22}\alpha_{21}+\overline{\gamma}_{23}\alpha_{31}+\overline{\gamma}_{23}\gamma_{21}\alpha_{32},\\ \delta_{31}&\equiv\overline{\gamma}_{32}\alpha_{21}+\overline{\gamma}_{33}\alpha_{31}+\overline{\gamma}_{33}\gamma_{21}\alpha_{32}.\end{aligned}$$

Setting $\alpha_{21}=0$, $\alpha_{31}=-\gamma_{21}\alpha_{32}$, then $\delta_{21}=\delta_{31}=0$ and $\delta_{23}=\gamma_{23}^2\alpha_{32}\neq 0$ if $\alpha_{32}\neq 0$.

Therefore, by Lemma III, the group H contains every binary transformation of determinant unity on ξ_2, ξ_3. Hence it contains

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & \mu\\ 0 & 0 & 1\end{pmatrix} \qquad (\mu \text{ arbitrary}),$$

and this, together with $B_{31,\lambda}$ $B_{32,\delta}$, generates a group of order p^6. But this contradicts the hypothesis on the order of H.

(III). By a reflexion* on the left diagonal we may draw the conclusion, with reference to the group G_{p^5} defined by

$$l_3\alpha_{320}+l_4\alpha_{321}=0,$$

that it can not contain any operation (β_{ij}) not contained in (3_3). H is then multiply isomorphic with the group of binary substitutions on ξ_1, ξ_2 of order pt (t prime to p).

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* Dickson, AMERICAN JOURNAL, Vol. XXVIII, p. 9.

We have now established the following theorem covering the results of the three cases I, II, III.

THEOREM. *Any subgroup H of order $p^5 N$ of the linear ternary homogeneous group G defined by the relation*

$$l_1 a_{210} + l_2 a_{211} + l_3 a_{320} + l_4 a_{331} = 0$$

(in which neither $l_1 = l_2 = 0$ nor $l_3 = l_4 = 0$) contains a self-conjugate subgroup G_{p^6}, and within G H is conjugate with a group of operations given by (3_1) *having* $\alpha_{22}^3 = 1$, $\alpha_{11}\alpha_{22}\alpha_{33} = 1$.

If H contains a subgroup G_{p^5} defined by the relation $l_1 a_{210} + l_2 a_{211} = 0$ or *$l_3 a_{320} + l_4 a_{331} = 0$, H is conjugate within G with a group of operations of the form* (3_1) or (3_2).

Subgroups of Order a Multiple of p^4.

8. Let H be a subgroup of order $p^4 N$ normalized to contain a subgroup G_{p^4} of G_{p^6}. First, let G_{p^4} be the special group $\{\alpha_{21} = 0\}$. We shall prove that it is self-conjugate under H.

If the number of conjugates to G_{p^4} were $1 + kp$ (k prime to p), then H would contain an operation P permutable with the G_{p^3} common to two G_{p^4} of the conjugate set to which G_{p^4} belongs, but not permutable with G_{p^4}, according to the Frobenius-Burnside Theorem. The operations permutable with the common G_{p^3} are, however, permutable with G_{p^4}. In the same way it follows that there can not be $1 + kp^2$ or $1 + kp^3$ conjugates to G_{p^4}. Finally, there can be no group H containing a G_{p^4} having $1 + kp^4$ conjugates. Indeed, by Lemma II, such a group H would be of index p^3 under G. But the linear homogeneous group G is isomorphic with the linear fractional $LF(3, p^2)$ and, if such an H existed, $LF(3, p^2)$ would be representable as a substitution group on p^2 letters and also its subgroup the linear fractional $LF(2, p^2)$. The latter can not be represented as a substitution on p^2 letters except when $p = 3$ (references in the Introduction). No such subgroup H can exist when $p = 3$, for the order of G when $p = 3$ is not a divisor of $p^2!$. Hence the subgroup $\{\alpha_{21} = 0\}$ is self-conjugate in H.

Next, by a reflexion on the left diagonal we reach the same conclusion with reference to $\{\alpha_{32} \equiv 0\}$.

Let H now be normalized to contain any one of the remaining groups of order p^4. If the number of conjugates to G_{p^4} is $1+kp^i$ (k prime to p, $i=1, 2, 3$), H must* contain an operation P permutable with $G_{p^{4-i}}$ but not with G_{p^4}.

First, let $i=1$. Then the greatest common subgroup is of order p^3, and the operation P is contained in one of the sets (3_1), (3_2), (3_3). We shall not consider (3_3), for the results obtained in case P lies in (3_2) are immediately applicable to the case when P lies in (3_3) by reflexion on the left diagonal. With this agreement the greatest common G_{p^3} of two groups of order p^4 is of one of two types:

$$\text{a)}\ \alpha_{21} \not\equiv 0, \quad \alpha_{32} \not\equiv 0;$$
$$\text{b)}\ \alpha_{21} \equiv 0, \quad \alpha_{32} \not\equiv 0.$$

The operations permutable with either of the groups a), b) are contained in the set (3_1) by Lemmas II and V, and P therefore is permutable with G_{p^4}, contradicting the Frobenius-Burnside Theorem. The number of conjugates to G_{p^4} under H can not be $1+kp$.

Next, let $i=2$ or 3. Then G_{p^m}, the greatest common subgroup, will be of order p^2 or p. P can not be contained in (3_1), since that leads to a contradiction of the Frobenius-Burnside Theorem. We consider P in the set (3_2). Any G_{p^4} of the set $(p, 3)$ contains the subgroup G_{p^2} defined by $\alpha_{21} \equiv \alpha_{32} \equiv 0$. The operations permutable with this G_{p^2} are contained in (3_1). Hence P in (3_2) is not permutable with G_{p^2}. It transforms G_{p^2} into G'_{p^2}; and $\{G_{p^2}, G'_{p^2}\}$ is of order p^4, G_{p^4} having $\alpha_{21} \equiv 0$. Hence $\{G'_{p^4} G_{p^4}\}$ is of order a power of p higher than p^4, and this contradicts the hypothesis on H. Hence the number of conjugates to G_{p^4} can not be $1+kp^2$ or $1+kp^3$. We have previously shown that the number can not be $1+kp^4$. Hence G_{p^4} must be self-conjugate under H, and we may state the following

THEOREM. *Every subgroup H, of order a multiple of p^4, of G contains a self-conjugate group G_{p^4}. Within G, H is conjugate with a group of operations* (3_2) *or a group of operations* (3_3).

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* Burnside: "Theory of Groups," p. 94, Cor. II, and the theorem on p. 97.

Groups of Transformations of Sylow Subgroups.

BY G. A. MILLER.

§ 1. *Introduction.*

The first part of the present paper is devoted to a study of some of the properties of a transitive substitution group which is a group of transformations of Sylow subgroups. In particular, we shall prove the following theorem: *If the Sylow subgroups of order* p^m *in any group* (G) *are transformed according to an imprimitive substitution group, the number of these subgroups must be of the form* $(1+k_1 p^\alpha)(1+k_2 p^\alpha)$, *where each of the two numbers* k_1, k_2 *exceeds zero and* α *is determined by the fact that the largest number of common operators in any two such Sylow subgroups of* G *is* $p^{m-\alpha}$. It is evident that α can not be less than unity nor greater than m.

In the second part of the paper we determine some properties of the groups which involve certain given numbers of Sylow subgroups. In particular, we prove that if a group G contains exactly three Sylow subgroups, these subgroups must generate a group of order $3 \,.\, 2^m$, and this group is a characteristic subgroup of G if it does not coincide with G; if G contains exactly four Sylow subgroups, these subgroups generate a group whose order is either $12 \,.\, 3^{m-1}$ or $24 \,.\, 3^{m-1}$, and this group is again either G itself or a characteristic subgroup of G. A somewhat similar theorem is proved as regards groups containing exactly six Sylow subgroups.

The $1+kp$ Sylow subgroups of order p^m in G are transformed according to a transitive substitution group (K) of degree $1+kp$ which has an $(1, \alpha)$-isomorphism with G. The invariant subgroup (H) of G which corresponds to the identity in K involves only one Sylow subgroup of order p^{α_1}, since the operators of H whose orders are powers of p must be contained in every Sylow subgroup of G. To every Sylow subgroup of G there corresponds one and only one Sylow subgroup of K, and hence G transforms its $1+kp$ Sylow subgroups of order p^m in exactly the same manner as K transforms its $1+kp$ Sylow subgroups of order p^β, $\beta \leqq m$. That is, the letters of the substitutions of K may stand for the various Sylow subgroups of order p^β in K.

The subgroup (K_1) of K which is composed of all its substitutions omitting one letter involves one and only one Sylow subgroup of order p^β and of degree kp. Moreover, if any transitive group has the property that its subgroup composed of all the substitutions which omit a given letter involves one and only one Sylow subgroup of degree one less than the degree of the group, it evidently permutes these Sylow subgroups in the same manner as it permutes its own letters.* Hence the theorem: *The necessary and sufficient condition that a transitive group of degree n is a group of transformations of Sylow subgroups is that its subgroup composed of all its substitutions which omit a given letter involves one and only one Sylow subgroup of degree* $n-1$.

From this theorem it follows directly that the only symmetric groups which are groups of transformations of Sylow subgroups are those of degrees 3 and 4, and the only alternating groups having this property are those of degrees 4 and 5. The symmetric group of degree 3 is clearly the only group of this degree which can be a group of transformations of Sylow subgroups, while the alternating and the symmetric group of degree 4 are the only two groups of degree 4 which have this property. There are three groups of degree 5 which are groups of transformations of Sylow subgroups; viz., one of each of the orders 10, 20, 60. Only two of the sixteen transitive groups of degree 6 are groups of transformations of Sylow subgroups; viz., the two groups of orders 60 and 120 respectively.

Since K_1 is of degree kp, it follows directly that a group of transformations of Sylow subgroups can not involve any invariant operator besides the identity.† From this fact and the elementary observations of the preceding paragraph it results that a large number of transitive substitution groups can not be groups of transformations of Sylow subgroups. In particular, it has been seen that no imprimitive group whose degree is less than eight can be such a group of transformations. We proceed to consider some of the properties of imprimitive groups which are also groups of transformations of Sylow subgroups.

§ 2. *Imprimitive Groups of Transformations of Sylow Subgroups.*

Suppose that K is imprimitive. All the substitutions of K_1 must transform at least one system of imprimitivity of K into itself. The number of letters in this system must be one more than the letters in one or more of the transitive constituents of K_1. Since each of these transitive constituents involves an invariant subgroup of order p^λ, $\lambda > 0$, it results that the degree of each of these constituents is divisible by p. In fact, if no two of the $1+kp$ Sylow subgroups of G have more than p^{m-a} common operators, each of these transitive constituents is divisible by p^a, and hence the number of letters in a system of imprimitivity of G is of the form $1+k_1p^a$.

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* *Bulletin of the American Mathematical Society*, Vol. II (1896), p. 145.

† Kuhn, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVI (1904), p. 67.

Since the degree of K is divisible by $1+k_1p^a$, it is necessary that the total number of letters in K is of the form $1+k_1p^a+k_0p^a$, where k_0 is divisible by $1+k_1p^a$, and hence the degree of K is $(1+k_1p^a)(1+k_2p^a)$, where each of the numbers k_1, k_2 exceeds 0. As K is any group of transformations of Sylow subgroups, we have proved the following theorem: *If the Sylow subgroups of order* p^m *in a group are transformed under this group according to an imprimitive substitution group, the number of these Sylow subgroups is of the form* $(1+k_1p^a)(1+k_2p^a)$, *where none of the integers* k_1, k_2, a *is zero.* The value of a may be determined by the fact that two of these Sylow subgroups have at most p^{m-a} operators in common.

When $p=2$, the theorem of the preceding paragraph does not yield any information, since every odd composite number can be represented in the form $(1+2k_1)(1+2k_2)$, where each of the numbers k_1, k_2 exceeds 0. Moreover, the dihedral group of order $2(1+2k_1)(1+2k_2)$ transforms its Sylow subgroups of order 2 according to an imprimitive group, k_1 and k_2 being any integers exceeding zero. It follows directly from the preceding theorem that 9 is the lowest degree for which imprimitive groups of transformations of Sylow subgroups exist, and it is not difficult to verify that there are exactly five groups of this degree which are imprimitive groups of transformations of Sylow subgroups; viz., two of order 18, one of order 36 and two of order 54.

Since the Sylow subgroups of order p^β in K generate a transitive group, it results that the operators in K_1 whose orders are powers of p must permute a multiple of p systems of imprimitivity of K. That is, the transitive group of degree $1+k_2p^a$ according to which the systems of imprimitivity of K are transformed is non-regular and involves substitutions of degree lp. In particular, not only must the Sylow subgroups always be transformed according to a non-regular group, but when these subgroups are transformed according to an imprimitive group these systems of imprimitivity must also be transformed according to a non-regular transitive group.

§ 3. *Groups Involving a Small Number of Sylow Subgroups of the Same Order.*

The smallest number of non-invariant Sylow subgroups of the same order in any group is evidently three, and if a group G contains exactly three Sylow subgroups of the same order this order is 2^m, and the operators of G transform these subgroups according to the symmetric group of order 6. To the identity of this symmetric group there corresponds a subgroup H of G which involves a characteristic subgroup of order 2^{m-1}. The subgroup of G which is generated by the three Sylow subgroups of order 2^m contains this characteristic subgroup, and the corresponding quotient group is generated by two operators of order 2. Hence this quotient group is dihedral and, as it involves only three operators of order 2, it must be the symmetric group of order 6. This proves the theorem: *If a group G involves exactly three Sylow subgroups of order 2^m, these subgroups generate a group of order $3 . 2^m$.* This group is evidently a characteristic subgroup of G if it does not coincide with G.

Suppose that G involves four Sylow subgroups of order 3^m and is generated by them. These four subgroups are transformed under G according to the alternating group of degree 4, since a group which contains a subgroup of half its order can not be generated by operators of odd order. The invariant subgroup H of G which corresponds to the identity in this alternating group involves a characteristic subgroup of order 3^{m-1}, and the corresponding quotient group (G') is generated by two operators (s_1, s_2) of order 3, and we may suppose that these operators correspond to the cycles abc, abd in the given alternating group.

Since $(s_1 s_2)^2$ is invariant under G', it results that $s_1 s_2$, $s_2 s_1$ are two operators which have a common square, and hence the product of one of these operators into the inverse of the other is transformed into its inverse by each of them. By means of this property it is easy to find an upper limit for the order of this product as follows:

$$(s_1 s_2 s_1^{-1} s_2^{-1})^3 = s_1 s_2 s_1^2 s_2^2 s_1 s_2 s_1^2 s_2^2 = (s_2 s_1)^2 s_1 s_2 s_1^2 s_2 s_1 s_2^2 = (s_1 s_2)^6.$$

Since $(s_1 s_2)^6$ is both invariant under K and also transformed into its inverse by $s_1 s_2$, it results that its order is a divisor of 12. This order can however not be divisible by 3, and hence it must divide 4. If $s_1 s_2$ is of order 2, it is clear that K is the tetrahedral group, as it is generated by two operators of order 3 whose product is of order 2. On the other hand, if $s_1 s_2$ is of order 4, the two operators $s_1 s_2$, $s_2 s_1$ must generate the quaternion group, since they have a common square and

$$(s_1 s_2)^{-1} s_2 s_1 . s_1 s_2 = s_2^2 s_1^2 s_2 s_1^2 s_2 = (s_2 s_1)^2 s_2^2 s_1 . s_2^2 s_1^2 s_1^2 s_2 = (s_2 s_1)^3,$$

since $s_1 s_2 = s_2^2 s_1^2 (s_2 s_1)^2$, and $(s_2 s_1)^2$ is invariant under G'. This quaternion group is invariant under G', since $s_1^2 s_2 s_1^2 = (s_2 s_1)^2 s_1 s_2^2 s_1 = (s_2 s_1)^2 . s_1 s_2 . s_2 s_1$. Hence

$\{s_1, s_2\}$ is the group of order 24 which does not involve a subgroup of order 12 whenever $s_1 s_2$ is of order 4.

These results establish the theorem: *If a group contains exactly four Sylow subgroups of the same order, they generate a group whose order is either* $4 \cdot 3^m$ *or* $8 \cdot 3^m$, *and the quotient group with respect to the* 3^{m-1} *common operators is either the tetrahedral group or the non-twelve group of order* 24. If the order of the entire group exceeds the order of the group generated by these Sylow subgroups, the latter group is clearly a characteristic subgroup of the former.

If G contains exactly six Sylow subgroups of order 5^m and is generated by these subgroups, it must transform them according to the primitive group of degree 6 and order 60. Hence G is isomorphic with the alternating group of degree 5. We shall first assume that $m = 1$ and that G is generated by the two operators s_1, s_2 which correspond to *abcde* and *aebcd* respectively of the isomorphic alternating group. Hence $s_1 s_2$ corresponds to the substitution of order 2, *ac . bd*. We shall prove that the order of $s_1 s_2$ is either 2 or 4. It is clear that the operators of G which correspond to the identity in the alternating group of order 60 constitute the central of G; and we shall prove that the order of this central is either 1 or 2, and hence the order of G is either 60 or 120.

Since $s_1 s_2$, $s_2 s_1$ have their squares in the central of G and are conjugate under G, they must have a common square, and hence the product of one of these operators into the square of the other is transformed into its inverse by each of them. By means of this property we obtain an upper limit for the order of one of the invariant operators of G as follows:

$$(s_1 s_2 s_1^4 s_2^4)^3 = s_1 s_2 s_1^4 s_2^4 s_1 s_2 s_1^4 s_2^4 s_1 s_2 s_1^4 s_2^4 = (s_1 s_2)^5 s_1^4 s_2^3 s_1^3 s_2^3 s_1^3 s_2^4 = (s_1 s_2)^6 (s_1^3 s_2^3)^3.$$

As $s_1^3 s_2^3$ corresponds to *abd*, and hence $(s_1^3 s_2^3)^3$ is in the central of G, it results that $(s_1 s_2)^6 (s_1^3 s_2^3)^3$ is invariant under G. It is also transformed into its inverse by $s_1 s_2$, and hence its order is 1 or 2. That is, $(s_1 s_2)^{12} = (s_2^2 s_1^2)^6$.

On the other hand, $s_1^3 s_2^2$ corresponds to *ae . bc*, and hence $(s_1^3 s_2^2)^2 = (s_2^2 s_1^3)^2$. Proceeding as above we obtain

$$\begin{aligned}(s_1^3 s_2^2 s_1^2 s_2^3)^3 &= s_1^3 s_2^2 s_1^2 s_2^3 s_1^3 s_2^2 s_1^2 s_2^3 s_1^3 s_2^2 s_1^2 s_2^3 = (s_1^2 s_2^2)^9 s_1 s_2^3 s_1^3 s_2 s_1 s_2^3 s_1^3 s_2 s_1 s_2^3 s_1^3 s_2 \\ &= (s_1^2 s_2^2)^9 (s_1 s_2)^5 (s_2^2 s_1^2)^3 s_1 s_2 = (s_1^2 s_2^2)^{12} (s_1 s_2)^6.\end{aligned}$$

From these equations it results that $(s_1 s_2)^{12} = (s_1^2 s_2^2)^{-24}$. Combining this with the earlier result, it follows that $(s_1^2 s_2^2)^{30} = 1$. As the order of $s_1^2 s_2^2$ can not be divisible by 5, it results that $(s_1^2 s_2^2)^6 = 1$ and hence $(s_1 s_2)^{12} = 1$.

It is evident that $(s_1 s_2)^2 = (s_1^4 s_2 s_1^2)^2$ and hence

$$\begin{aligned}(s_1 s_2 s_1^3 s_2^4 s_1)^5 &= s_1 s_2 s_1^3 s_2^4 s_1^2 s_2 s_1^3 s_2^4 s_1^2 s_2 s_1^3 s_2^4 s_1^2 s_2 s_1^3 s_2^4 s_1^2 s_2 s_1^3 s_2^4 s_1 \\ &= (s_1 s_2)^9 s_1^3 s_2^4 s_1 s_2^4 s_1^2 s_2^4 s_1 s_2^4 s_1^2 s_2^4 s_1 s_2^4 s_1^2 s_2^4 s_1 s_2^4 s_1^2 s_2^4 s_1 \\ &= (s_1^2 s_2^3)^8 (s_1 s_2)^9 s_1^3 s_2^4 s_1 s_2 s_1^3 s_2 s_1 s_2 s_1^3 s_2 s_1 s_2 s_1^3 s_2 s_1 s_2 s_1^3 s_2 s_1 \\ &= (s_1^2 s_2^3)^8 (s_1 s_2)^{17} s_1^3 s_2^3 s_1^3 s_2 s_1 = (s_1^2 s_2^3)^8 (s_1 s_2)^{19} s_1^3 s_2^3 s_1^2 s_2^4 \\ &= (s_1^2 s_2^3)^{10} (s_1 s_2)^{19} s_1^3 \,.\, s_1^3 s_2^2 \,.\, s_2^4 = (s_1^2 s_2^3)^{10} (s_1 s_2)^{20}.\end{aligned}$$

From these equations it results that $(s_1 s_2)^{40} = (s_2^2 s_1^3)^{20}$, or $(s_1 s_2)^4 = (s_2^2 s_1^3)^{20}$, since $(s_1 s_2)^{12} = 1$.

As $(s_1 s_2)^3 = (s_2^3 s_1 s_2^3)^2$, we obtain in a similar manner the following equations:

$$\begin{aligned}(s_1 s_2^3 s_1^4 s_2^2)^5 &= s_1 s_2^3 s_1^4 s_2^2 s_1 s_2^3 s_1^4 s_2^2 s_1 s_2^3 s_1^4 s_2^2 s_1 s_2^3 s_1^4 s_2^2 s_1 s_2^3 s_1^4 s_2^2 \\ &= (s_1 s_2)^9 s_2^2 s_1^4 s_2 s_1^4 s_2^2 s_1^4 s_2 s_1^4 s_2^2 s_1^4 s_2 s_1^4 s_2^2 s_1^4 s_2 s_1^4 s_2^2 s_1^4 s_2^2 \\ &= (s_2^2 s_1^3)^8 (s_1 s_2)^9 s_2^2 s_1^4 s_2 s_1 s_2^3 s_1 s_2 s_1 s_2^3 s_1 s_2 s_1 s_2^3 s_1 s_2 s_1 s_2^3 s_1 s_2^2 \\ &= (s_2^2 s_1^3)^8 (s_1 s_2)^{17} s_2^2 s_1^3 s_2^3 s_1 s_2^2 = (s_2^2 s_1^3)^{10} (s_1 s_2)^{20}.\end{aligned}$$

From these equations it follows directly that $(s_1 s_2)^{40} = (s_2^2 s_1^3)^{-20}$, and if we combine this with the result of the preceding paragraph, we have that $(s_2^2 s_1^3)^{40} = 1$. Since the order of $s_2^2 s_1^3$ is not divisible by 5, it results that $(s_2^2 s_1^3)^8 = 1$, and hence $(s_1 s_2)^{80} = 1$. If the last equation be combined with $(s_1 s_2)^{12} = 1$, it results that $(s_1 s_2)^4 = 1 = (s_1^2 s_2^3)^4 = (s_2^2 s_1^3)^4$.

Since

$$\begin{aligned}(s_2^2 s_1^3)^2 &= s_2^2 s_1^3 s_2^2 s_1^3 = s_2^2 s_1^2 \,.\, s_1 s_2 \,.\, s_2 s_1 \,.\, s_1^2 = s_2^2 s_1^2 s_2^4 s_1^3 s_2^4 s_1^2 \\ &= s_2^2 s_1^2 s_2^2 \,.\, s_2^2 s_1^3 \,.\, s_2^4 s_1^2 = s_2^2 s_1^2 s_2^2 s_1^2 s_2^2 s_1^2 (s_2^2 s_1^3)^2,\end{aligned}$$

it results that $(s_1^2 s_2^3)^3 = 1$. That is, $s_2^2 s_1^3$, $s_1^2 s_2^3$ are two operators of orders 4, 3 or 2, 3 respectively, whose product is of order 5, and the square of the former is invariant under both of these operators. Hence they must generate either the icosahedral group or the non-sixty group of order 120 which is isomorphic with this group. It is moreover clear that each of these two groups can be generated by two of its operators of order 5.

When $m > 1$, the six Sylow subgroups of order 5^m in G have 5^{m-1} common operators, and the corresponding quotient group has six Sylow subgroups of order 5 and is generated by them. In fact, if a group has exactly $p + 1$ Sylow subgroups of order p^m and is generated by them, these subgroups have p^{m-1} common operators which form an invariant subgroup, and the corresponding quotient group involves exactly $p + 1$ Sylow subgroups of order p and is generated by them. The preceding results clearly establish the theorem: *If a group involves exactly six Sylow subgroups of order* 5^m, *these subgroups generate a group whose order is either* $12 \,.\, 5^m$ *or* $24 \,.\, 5^m$.

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q-Difference Equations.

By Rev. F. H. Jackson.

§ 1.

The properties of the q-hypergeometric series

$$1+\frac{(1-q^{\alpha})(1-q^{\beta})}{(1-q\ )(1-q^{\gamma})}x+\frac{(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})}{(1-q\ )(1-q^{2}\ )(1-q^{\gamma})(1-q^{\gamma+1})}x^{2}+\dots$$

have been investigated by Heine (*Kugelfunctionen*, ed. 1878, pp. 99 et seq.); also by Thomæ (*Crelle's Journal*, Vol. LXX). In this note I wish to introduce a notation which will, I hope, be of utility in developing the theory of q-functions generally: For example, if we replace the differential operative symbol ϑ in the differential equation of the hypergeometric series $F(\alpha, \beta, \gamma, x)$ by

$$[\vartheta]=(q^{\vartheta}-1)/(q-1),$$

we have

$$x[\vartheta+\alpha][\vartheta+\beta]y-[\vartheta][\vartheta+\gamma-1]y=0, \tag{1}$$

a q-difference equation with particular solutions corresponding in every detail to the solutions of

$$x(\vartheta+\alpha)(\vartheta+\beta)y-\vartheta(\vartheta+\gamma-1)y=0.$$

If we define Δ, a symbol of operation, by the equation

$$\Delta\Phi(x)=\frac{\Phi(qx)-\Phi(x)}{qx-x}, \tag{2}$$

then symbolically, in terms of the differential operator $\left(\vartheta=x\dfrac{d}{dx}\right)$,

$$\begin{aligned} x\Delta &= \frac{q^{\vartheta}-1}{q\ -1} \\ &= [\vartheta]. \end{aligned}$$

Repetition of (2) gives

$$\Delta^{2}\Phi(x)=\frac{\Phi(q^{2}x)-(q+1)\Phi(qx)+q\Phi(x)}{qx^{2}(q-1)^{2}},$$

which is symbolically

$$x^2\Delta^2 = \frac{q^{2\vartheta} - (q+1)q^{\vartheta} + q}{q(q-1)^2} \qquad x^2\Delta^2 = [\vartheta][\vartheta - 1],$$

and in general,

$$x^n\Delta^n = [\vartheta][\vartheta - 1] \dots [\vartheta - n + 1]. \tag{3}$$

We see that in case the functions on which we operate are differentiable in the range of the variable, we may write

$$\operatorname*{limit}_{q \longrightarrow 1} [\vartheta] = \vartheta = x\frac{d}{dx},$$

$$\operatorname*{limit}_{q \longrightarrow 1} [\vartheta][\vartheta - 1] \dots [\vartheta - n + 1]$$
$$= \vartheta(\vartheta - 1) \dots (\vartheta - n + 1) = x^n \frac{d^n}{dx^n} \quad \text{(Boole)}. \tag{4}$$

§ 2.

The well-known and very useful operational theorem

$$F(\vartheta)x^m v = x^m F(\vartheta + m)v \tag{5}$$

has its counterpart in

$$F[\vartheta]x^m v = x^m F[\vartheta + m]v\,;$$

for

$$\begin{aligned} [\vartheta]x^m v_x &= (q^m x^m v_{qx} - x^m v_x)/(q-1) \\ &= x^m(q^m v_{qx} - v_x)/(q-1) \\ &= x^m(q^{m+\vartheta}v - v)/(q-1) \\ &= x^m[\vartheta + m]v\,; \end{aligned}$$

therefore

$$[\vartheta]^r x^m v = x^m[\vartheta + m]^r v.$$

This equation holds good when r is a negative integer; for, on writing

$$[\vartheta + m]^r v = V, \qquad v = [\vartheta + m]^{-r} V,$$

we have

$$[\vartheta]^r x^m [\vartheta + m]^{-r} V = x^m V;$$

on operating with $[\vartheta]^{-r}$, we obtain

$$x^m[\vartheta + m]^{-r} V = [\vartheta]^{-r} x^m V;$$

so that, finally, if $F(a)$ denotes an algebraic function of a, we have symbolically

$$F([\vartheta])x^m v = x^m F([\vartheta + m])v.$$

(COROLLARY). It will follow that if U be any function of the form $A + Bx + Cx^2 + \dots,$

$$\frac{1}{\psi[\vartheta]}U = \frac{A}{\psi[0]} + \frac{B}{\psi[1]}x + \frac{C}{\psi[2]}x^2 + \dots \tag{6}$$

§ 3.

Before proceeding to solve any q-difference equations, I remark that if the reverse operation Δ^{-1} be named q-integration, and we use the symbolism

$$\Delta^{-1}\phi(x) \equiv \mathop{\mathrm{S}} \Phi(x)d(qx), \tag{7}$$

remembering that subject to conditions of integrability

$$\operatorname*{limit}_{q \longrightarrow 1} \mathop{\mathrm{S}} \Phi(x)d(qx) = \int \Phi(x)dx, \tag{8}$$

then the q-hypergeometric difference equation will have a solution

$$\mathop{\mathrm{S}}\limits_0^1 u^{\beta-1}(1-qu)^{\gamma-\beta-1}(1-q^{\alpha}ux)^{-\alpha}d(qu)$$

reducing in the limit ($q \longrightarrow 1$) to

$$\int u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-ux)^{-\alpha}du.$$

Similarly, for the q-series

$$F_q([\alpha][\beta][\gamma][\delta][\varepsilon]; x),$$

and equation

$$x[\vartheta+\alpha][\vartheta+\beta][\vartheta+\gamma]y - [\vartheta][\vartheta+\delta-1][\vartheta+\varepsilon-1]y = 0,$$

we have an integral

$$\mathop{\mathrm{S}}\limits_0^1 u^{\beta-1}(1-qu)^{\delta-\beta-1}v^{\gamma-1}(1-qu)^{\epsilon-\gamma-1}(1-q^{\alpha}uvx)^{-\alpha}d(qu)d(qv).$$

Of course $(1-qu)^m$ denotes the q-binomial product

$$\prod_{r=1}^{\infty} \frac{(1-q^r u)}{(1-q^{n+r}u)} = 1 - q[n]u + q^3\frac{[n][n-1]}{[2]!}u^2 - \dots \qquad (|q|<1).$$

§ 4.

q-Finite Integration.

It is well-known (Bromwich, "Theory of Infinite Series," pp. 418–419)

$$\int_0^b f(x)dx = \operatorname*{limit}_{q \longrightarrow 1} b(1-q)\{f(b) + qf(bq) + q^2 f(bq^2) + \dots\}.$$

Consider

$$\Delta u = \phi(x),$$

which is

$$\frac{1}{x}[\vartheta]u = \phi(x),$$

$$\begin{aligned} u &= \frac{(1-q)}{(1-q^{\vartheta})}x\phi(x) + \frac{(1-q)}{(1-q^{\vartheta})}\cdot 0 \\ &= (1-q)\{1 + q^{\vartheta} + q^{2\vartheta} + \cdots\}x\phi(x) + C, \end{aligned}$$

$$\mathop{\mathrm{S}}\limits_{0}^{b}\phi(x)d(qx) = b(1-q\{\Phi(b) + q\Phi(qb) + q^2\Phi(q^2b) + \cdots\}. \qquad (9)$$

C will be in general any function of multiplicative periodicity q. Such a function, for example, as

$$exp\left(i\frac{\log x}{\log q}\right).$$

If in (9) we expand $(1-q)/(1-q^{\vartheta})$ in powers of ϑ, using Bernoulli's coefficients, we find

$$\begin{aligned} \mathop{\mathrm{S}}\limits_{0}^{b}\Phi(x)d(qx) = &\frac{(q-1)}{\log q}\int_0^b \phi(x)dx + \tfrac{1}{2}(1-q)b\Phi(b) \\ &+ (1-q)\sum_{r=1}^{\infty}(-1)^r\frac{B_{2r-1}}{2r\,!}(\log q)^{2r-1}(\vartheta)^{2r-1}x\phi(x)\Big]_{x=b}. \end{aligned} \qquad (10)$$

Now $\lim\limits_{q=1}\frac{q-1}{\log q} = 1$, so that in this form we see in detail how

$$\int_0^b \Phi(x)dx = \lim_{q\longrightarrow 1} b(1-q)\{\Phi(b) + q\Phi(qb) + \cdots\}. \qquad (11)$$

§ 5.

Solutions of Equations in Series.

Since

$$[\vartheta + \alpha]x^n = \frac{q^{\vartheta+\alpha}-1}{q-1}x^n = \frac{q^{\alpha}q^{\vartheta}x^n - x^n}{q-1} = \frac{(q^{\alpha+n}-1)x^n}{(q-1)} = [\alpha + n]x^n,$$

$$[\vartheta + \alpha][\vartheta + \beta]x^n = [n + \alpha][n + \beta]x^n$$

and

$$\begin{aligned} [\vartheta + \alpha]x^n \log x &= \frac{q^{\vartheta+\alpha}x^n \log x - x^n \log x}{q-1} \\ &= \frac{q^{\alpha}q^{n}x^n \log qx - x^n \log x}{q-1} = [\alpha + n]x^n \log x + \frac{\log q}{q-1}x^n q^{\alpha+n}, \end{aligned}$$

$$\begin{aligned} [\vartheta + \alpha][\vartheta + \beta]x^n \log x = &[\alpha + n][\beta + n]x^n \log x + \lambda q^{\alpha+n}[\beta + n]x^n \\ &+ \lambda q^{\beta+n}[\alpha + n]x^n \qquad (\lambda = \log q/(q-1)), \end{aligned} \qquad (12)$$

it is obvious that if we substitute the series

$$A_0x^\theta + A_1x^{\theta+1} + A_2x^{\theta+2} + \ldots.$$

for y in the equation

$$x[\vartheta + \alpha][\vartheta + \beta]y - [\vartheta][\vartheta + \gamma - 1]y = 0,$$

we obtain an indicial equation

$$[\theta][\theta + \gamma - 1] = 0$$

with two principal roots $\theta = 0,\ 1 - \gamma$, and the usual infinity of roots involving the logarithm. Taking

$$A_n = A_0 \frac{[n - 1 + \alpha + \theta] \ldots . [\alpha] \,.\, [n - 1 + \beta + \theta] \ldots . [\beta]}{[n + \theta] \ldots . [1 + \theta] \,.\, [n - 1 + \gamma + \theta] \ldots . [\gamma]},$$

we find for the root $\theta = 0$, the solution

$$F_q([\alpha][\beta][\gamma], x),$$

and for the root $\theta = 1 - \gamma$ a particular solution

$$x^{1-\gamma}F_q([\alpha - \gamma + 1][\beta - \gamma + 1]\,[2 - \gamma];\, x).$$

In case $1 - \gamma$ be a positive integer, say p, we substitute for y the series

$$\sum_{r=0}^{\infty}(A_r + B_r \log x)x^{r+\theta}.$$

Now if $\Pi_q[m]$ denote the q-Π function analogous to Gauss' Π function,

$$\Pi_q[m] = [m]\Pi_q[m - 1].$$

If $\dfrac{d}{dx}\log \Pi_q([m])$* be denoted Ψ_m, we have

$$\Psi_m = \lambda \frac{q^m}{[m]} + \Psi_{m-1},$$

where $\lambda = \log q/(q - 1)$. The substitution of

$$\sum(A_r + B_r \log x)x^{\theta+r} \quad \text{in} \quad x[\alpha + \vartheta][\beta + \vartheta]y - [\vartheta][\vartheta - p]y = 0$$

gives an indicial equation $[\theta][\theta - p] = 0$; and if $\sum$ is to be a solution,

$$\begin{aligned}\sum_{r=0}^{\infty}\{A_r[\alpha + \theta + r][\beta + \theta + r]x^{\theta+r+1} &- A_r[\theta + r][\theta - p + r]x^{\theta+r},\\ B_r[\alpha + \theta + r][\beta + \theta + r]x^{\theta+r+1}\log x &- B_r[\theta + r][\theta - p + r]x^{\theta+r},\\ \lambda B_r[\alpha + \theta + r]q^{\beta+\theta+r}x^{\theta+r+1} &- \lambda B_r[\theta + r]q^{\theta+r-p}x^{\theta+r},\\ \lambda B_r[\beta + \theta + r]q^{\alpha+\theta+r}x^{\theta+r+1} &- \lambda B_r[\theta - p + r]q^{\theta+r}x^{\theta+r}\}\end{aligned}$$

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* *Proc. R. S. Lond.*, Vol. LXXVI A, p. 180.

must vanish identically. Hence

$$\begin{aligned} &A_{r+1}[\theta + r + 1][\theta - p + r + 1] \\ &\qquad = A_r[\theta + \alpha + r][\theta + \beta + r] \qquad (r = 0, 1, \ldots, p - 1), \\ &A_p,\ A_{p+1}, \ldots = 0. \end{aligned}$$

So putting $\theta = 0$, and writing $[\alpha]\,[\alpha + 1] \ldots [\alpha + n - 1] = [\alpha]_n$, we find a solution

$$\begin{aligned} &\frac{[\alpha]_p[\beta]_p}{[p]!\,[1 - p][2 - p] \ldots [-1]} x^p F_q([\alpha + p][\beta + p][1 + p], x) \log x \\ &+ \sum_{n=0}^{p=1} \frac{[\alpha] \ldots [\alpha + n - 1] \,.\, [\beta] \ldots [\beta + n - 1]}{[n]!\,[1 - p][2 - p] \ldots [n - p]} x^n \\ &+ \sum_{n=p}^{\infty} \frac{[\alpha]_n[\beta]_n}{[n]!\,[n - p]!\,[1 - p][2 - p] \,.\, [-1]} x^n \Phi_n, \end{aligned} \tag{13}$$

where $\Phi_n = \Psi[n - 1 + \alpha) + \Psi[n - 1 + \beta] - \Psi[n] - \Psi[n - p]$. We see that

$$\Phi_n - \Phi_{n-1} = \frac{q^{n-1+\alpha}}{[n - 1 + \alpha]} + \frac{q^{n-1+\beta}}{[n - 1 + \beta]} - \frac{q^n}{[n]} - \frac{q^{n-p}}{[n - p]}.$$

The solution, in case $1 - \gamma$ be a negative integer, offers no difficulty, and in fact we can use a slight modification of Frobenius' method to obtain these forms of integrals. Simply substituting $\sum c_r x^{\theta+r}$ in the q-difference equation, choosing

$$c_n = c_0 \frac{[n - 1 + \alpha + \theta] \ldots [\alpha][n - 1 + \beta + \theta] \ldots [\beta]}{[n + \theta] \,.\, [1 + \theta] \,.\, [n - p + \theta] \ldots [1 - p + \theta]},$$

make $c_0 = C[\theta]$, so that

$$y\phi(\vartheta)y = C[\theta]^2[\theta - p]x^\theta,$$

$$y = C \sum_{n=0} \frac{[\alpha + \theta]_n[\beta + \theta]_n}{[1 + \theta]_n[1 - p + \theta]_n} [\theta] x^{\theta+n}.$$

Two particular integrals will be given by

$$\{y\}_{\theta=0} \quad \text{and} \quad \left\{\frac{dy}{d\theta}\right\}_{\theta=0}.$$

The first gives a constant multiple of $x^p F_q([\alpha + p][\beta + p][1 + p], x)$. The second, after making $C = 1$, reduces to (13), given above. In case $1 - \gamma$ is a negative integer, $-p$, we write $c_0 = [\theta + p]C$ and find two particular solutions given by

$$\{y\}_{\theta=-p}; \qquad \left\{\frac{dy}{d\theta}\right\}_{\theta=-p}.$$

The first of these gives $F_q([\alpha][\beta][1+p], x)$; the second gives

$$\frac{[\alpha-1]\,.\,[\alpha-p]\,.\,[\beta-1]\,..\,[\beta-p]}{[p]\,!\,[1-p]\,..\,[-1]}\,F_q([\alpha][\beta][1+p], x)\log x$$

$$+\sum_{n=0}^{p-1}\frac{[\alpha-p]_n[\beta-p]_n}{[n]\,!\,[n-q]\,.\,[1-q]}\,x^{n-p}$$

$$+\sum_{n=p}^{\infty}\frac{[\alpha-p]_n[\beta-p]_n}{[n]\,!\,[n-q]\,!\,[1-q][2-p]\,.\,[-1]}\,x^{n-p}\Phi_n, \tag{14}$$

$$\Phi_n=\psi[n-1+\alpha-p]+\psi[n-1+\beta-p]-\psi[n]-\psi[n-p].$$

§ 6.

The General q-Hypergeometric Series.

$$F_q([\alpha_1]\,\ldots\,[\alpha_r],\,[\beta_1]\,[\beta_2]\,\ldots\,[\beta_s],\,x),$$

$$x^{1-\beta_m}F_q[(\alpha_1-\beta_m+1]\ldots[\alpha_r-\beta_m+1],$$
$$[2-\beta_m][\beta_1-\beta_m+1]\,..\,[\beta_s-\beta_m+1],\,x),$$

where $m=1, 2, 3, 4, \ldots, s$ and $1-\beta_m$ is not zero or an integer, are solutions of

$$x[\vartheta+\alpha_1][\vartheta+\alpha_2]\,..\,[\vartheta+\alpha_r]y-[\vartheta][\vartheta+\beta_1-1]\,..\,[\vartheta+\beta_s-1]y=0. \tag{15}$$

§ 7.

Solution of $[\vartheta+k]\,y=V$ *(*V *a given function of* x*).*

We transform this equation into

$$y=[\vartheta+k]^{-1}V,$$

which by (5) becomes

$$y=x^{-k}[\vartheta]^{-1}x^kV.$$

Expanding $[\vartheta]^{-1}$ in powers of ϑ,

$$y=C+\frac{(q-1)}{\log q}x^{-k}\int Vx^{k-1}dx-(q-1)\frac{V}{2}$$
$$+(q-1)\sum_{r=1}^{\infty}(-1)^{r-1}\frac{B_{2r-1}}{2r\,!}(\log q)^{2r-1}x^{-k}\Big(x\frac{d}{dx}\Big)^{r-1}x^kV.$$

The solution of

$$[\vartheta+\alpha_1][\vartheta+\alpha_2]\ldots[\vartheta+\alpha_n]y=V \tag{16}$$

by resolution of the inverse operator $\{[\vartheta+\alpha_1]\ldots[\vartheta+\alpha_n]\}^{-1}$ into partial fractions is obvious. Now an equation

$$a_2\Phi(q^2x)+a_1\Phi(qx)+a_0\Phi(x)=V,$$

in which V is a given function of x and the form Φ is to be determined, is equivalent to finding a solution of

$$[\vartheta + \alpha][\vartheta + \beta]y = V.$$

In general, such an equation as (16), given above, corresponds to

$$a_n\Phi(q^n x) + a_{n-1}\Phi(q^{n-1}x) + \ldots\ldots + a_1\Phi(qx) + a\Phi(x) = V,$$

and Φ can be determined provided the factors of the polynomial

$$a_n r^n + a_{n-1} r^{n-1} + \ldots\ldots + a_1 r + a_0$$

are known.

§ 8.

Definite Integral Solutions.

To determine a q-finite integral form satisfying the difference equations of the q-hypergeometric series we proceed as follows: Let

$$(1 - au)^{(n)} = \prod_{r=0}^{\infty} \frac{(1 - auq^r)}{(1 - auq^{r+n})},$$

we note that

$$\Delta(1 - au)^{(n)} = - a[n](1 - qau)^{(n-1)},$$

so that, using the formula for q-part integration, we have

$$\int u^{m-1}(1 - qu)^{(n-1)} = - \frac{1}{[n]} u^{m-1}(1 - u)^{(n)} + \int \frac{[m-1]}{[n]} u^{m-2}(1 - qu)^{(n)}.$$

Now in case n be positive, $(1 - u)^{(n)}$ vanishes when $u = 1$, so that we have

$$\int_0^1 u^{m-1}(1 - qu)^{(n-1)} = \frac{[m-1]}{[n]} \cdot \int_0^1 u^{m-2}(1 - qu)^{(n)},$$

which serves to indicate a function $B_q([m], [n])$ corresponding to Euler's Beta Function. If we expand $(1 - qu)^{(n-1)}$ as

$$1 - [n-1]qu + \frac{[n-1][n-2]}{[2]!} q^2u^2 + \ldots\ldots + (-1)^r \frac{[n-1]\ldots\ldots[n-r]}{[r]!} q^{\frac{1}{2}r(r+1)}u^r + \ldots\ldots$$

and take the q-integral

$$\int_0^1 u^{m-1}\{1 - [n-1]qu + \ldots\ldots\} d(qu)$$

term by term, we get

$$\frac{1}{[m]}\left\{1 - \frac{[n-1][m]}{[m+1][1]} q + \frac{[n-1][n-2][m][m+1]}{[m+1][m+2][2]!} q^2 - \ldots\ldots\right\}$$

$$= \frac{1}{[m]}\left\{1 + \frac{[1-n][m]}{[m+1][1]} q^n + \frac{[1-n][2-n][m][m+1]}{[m+1][m+2][2]!} q^{2n} + \ldots\ldots\right\},$$

$$= \frac{1}{[m]} \prod_{r=1}^{\infty} \frac{[m+n+r]}{[n+r][m+r+1]}$$

$$= \prod_{r=1}^{\infty} \frac{[m+n+r]}{[n+r][m+r]}$$

$$= \frac{\Gamma_q(m)\Gamma_q(n)}{\Gamma_q(m+n)},$$

where $\Gamma_q(m)$ is the basic or q-gamma function* whose difference equation is

$$\Gamma_q(n+1) = [n]\Gamma_q(n).$$

The above is a special case (for $\alpha = 1 - n$, $\beta = m$, $\gamma = m + 1$) of the well-known q-hypergeometric series

$$F([\alpha]\,[\beta]\,[\gamma]q^{\gamma-\alpha-\beta}),$$

the sum of which is

$$\frac{\Gamma_q(\gamma-\alpha-\beta)\Gamma_q(\gamma)}{\Gamma_q(\gamma-\alpha)\Gamma_q(\gamma-\beta)}$$

If now we consider

$$\mathop{S}\limits_0^1 u^{\beta-1}(1-qu)^{\gamma-\beta-1}(1-q^{\alpha}xu)^{-\alpha}d(qu)$$

and integrate after replacing $(1-q^{\alpha}xu)^{-\alpha}$ by its expansion

$$1 + [\alpha]x + \frac{[\alpha][\alpha+1]}{[2]\,!}x^2 + \dots,$$

we find that this integral is

$$\frac{\Gamma_q(\beta)\Gamma_q(\gamma-\beta)}{\Gamma_q(\gamma)} F_q([\alpha][\beta][\gamma], x), \tag{17}$$

provided β and $\gamma - \beta$ are both positive.

Similarly, the double q-integral

$$\mathop{S}\limits_0^1 \mathop{S}\limits_0^1 u^{\beta-1}(1-qu)^{\delta-\beta-1}v^{\gamma-1}(1-qv)^{\epsilon-\gamma-1}(1-q^{\alpha}uvx)^{-\alpha}d(qu),$$

if we expand $(1-q^{\alpha}uv)^{-\alpha}$ and integrate term by term the series which is absolutely convergent if $|q| < 1$, becomes

$$\mathop{S}\limits_0^1 u^{\beta-1}(1-qu)^{\delta-\beta-1}\frac{\Gamma_q(\gamma)\Gamma_q(\varepsilon-\gamma)}{\Gamma_q(\varepsilon)} F_q([\alpha][\gamma][\varepsilon], ux)d(qu). \tag{18}$$

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* *Proc. R. S. Lond.*, Vol. LXXVI A, p. 180.

Replacing $F_q([\alpha][\gamma][\varepsilon], ux)$ by the infinite series and taking the q-integrals term by term, we have for the general term

$$\frac{\Gamma_q(\gamma)\Gamma_q(\varepsilon-\gamma)}{\Gamma_q(\varepsilon)} \int_0^1 \frac{[\alpha]_r[\gamma]_r}{[\varepsilon]_r} x^r u^{\beta+r-1}(1-qu)^{\delta-\beta-1} d(qu),$$

$$= \frac{\Gamma_q(\gamma)\Gamma_q(\varepsilon-\gamma)[\alpha]\ldots\ldots[\alpha+r-1][\gamma]\ldots[\gamma+r-1]}{\Gamma_q(\varepsilon)[\varepsilon]\ldots\ldots[\varepsilon+r-1]} \times \frac{\Gamma_q(\beta+r)\Gamma_q(\delta-\beta)}{\Gamma_q(\delta+r)} x^r;$$

so that (18)

$$= \frac{\Gamma_q(\gamma)\Gamma_q(\varepsilon-\gamma)\Gamma_q(\beta)\Gamma_q(\delta-\beta)}{\Gamma_q(\varepsilon)\Gamma_q(\delta)} F_q([\alpha][\beta][\gamma], [\delta][\varepsilon], x),$$

provided β, $\delta-\beta$, γ, $\varepsilon-\gamma$ be all positive quantities.

§ 9.

In conclusion, it may be of interest to note that

$$[\vartheta+n][\vartheta-n]y + x^2y = 0$$

is the q-equation satisfied by a function

$$J_n(q, x) = \frac{x^n}{[2][4]\ldots[2n]}\left\{1 - \frac{x^2}{[2][2n+2]} + \frac{x^4}{[2][4][2n+2][2n+4]} - \ldots\right\}.$$

In case n be not a positive integer, $[2][4]\ldots[2n]$ may be replaced by a suitable q-gamma function.

$$[\vartheta+m+n][\vartheta+m-n][\vartheta-m+n][\vartheta-m-n]y + [2\vartheta][2\vartheta-2]x^2y = 0$$

is the equation satisfied by

$$J_m(q, x) \times J_n\left(\frac{1}{q}, x\right)$$

$$= \sum_{r=0}^{r=\infty} (-1)^r \frac{\{2m+2n+4r\}!}{\{2m+2n+2r\}!\{2m+2r\}!\{2n+2r\}!\{2r\}!} x^{m+n+2r}.$$

The theory of these functions, apart from the q-difference equations which they satisfy, has been discussed (*Proc. L. M. S.*, Ser. 2, Vol. II, Part III; Vol. III, Part I).

We note that

$$\{2m\}! = [2][4]\ldots[2m] \text{ and } J_n\left(\frac{1}{q}, x\right) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{n+2r}}{\{2n+2r\}!\{2r\}!} q^{n^2+2r(n+r)}.$$

On the Relation between the Sum-Formulas of Hölder and Cesàro.

By Walter B. Ford.

I. *Introduction.*

1. If

$$u_0 + u_1 + u_2 + \dots + u_n + \dots \tag{1}$$

be a series (convergent or divergent) such that for some positive integral value of r the expression $\lim_{n=\infty} S_n^{(r)}/D_n^{(r)}$ exists and has the value l, where it is understood that

$$\left.\begin{aligned} S_n^{(r)} &= s_n + rs_{n-1} + \frac{r(r+1)}{2!}s_{n-2} + \dots \frac{r(r+1)\dots(r+n-1)}{n!}s_0, \\ s_n &= u_0 + u_1 + u_2 + \dots + u_n, \\ D_n^{(r)} &= \frac{(r+1)(r+2)\dots(r+n)}{n!}, \end{aligned}\right\} \tag{2}$$

then the series is said to be summable for the given value of r in Cesàro's sense, and to have the sum l.* The same series is said to be summable for a given value of r in Hölder's sense, and to have the sum l when $\lim_{n=\infty} s_n^{(r)} = l$, where $s_n^{(0)} = s_n$ and

$$s_n^{(r)} = \frac{1}{n+1}(s_0^{(r-1)} + s_1^{(r-1)} + s_2^{(r-1)} + \dots + s_n^{(r-1)}), \tag{3}$$

it being understood that s_n has the meaning already indicated.†

In the following paper it is proposed to study the relation between the sums l of (1) as given by the two preceding formulas. In this connection we shall establish the following theorem, showing that the formulas are coextensive in applicability and in their resulting sums.

Theorem. *If the series* (1) *is summable for a specified value of r by Hölder's formula, it is summable for this value of r by Cesàro's formula, and the two sums*

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* It may be remarked that the value of the limit l is the same for all values of r for which it exists.

† Cf. Bromwich, *Infinite Series* (1908), §§ 122, 128.

are identical. Conversely, if the series (1) *is summable for a given value of r by Cesàro's formula, it is summable for this value of r by Hölder's formula, and the two sums are identical.**

II. *Reduction of the Problem.*

2. We shall first consider the existence of the above theorem when the expressions $s_n^{(0)}, s_n^{(1)}, s_n^{(2)}, \ldots, s_n^{(r)}$, instead of being defined by (3), are defined by the following analogous system of relations:

$$s_n^{(0)} = s_n; \quad s_n^{(r)} = \frac{1}{n+r}\left(s_0^{(r-1)} + s_1^{(r-1)} + \ldots + s_n^{(r-1)}\right). \tag{4}$$

To do this it is desirable for our purpose to at once determine the n coefficients $a_0(n, r)$, $a_1(n, r)$, $a_2(n, r)$, ..., $a_n(n, r)$ (each being a function of both n and r) such that for all values (positive integral) of n and r we shall have

$$S_n^{(r)} = a_0(n, r)\, s_n^{(r)} + a_1(n, r)\, s_{n-1}^{(r)} + \ldots + a_n(n, r)\, s_0^{(r)}. \tag{5}$$

Assuming for the moment the existence of such coefficients, let us suppose n to have some *fixed* value. From (2) we have

$$s_n^{(r)} = (n + r + 1)\, s_n^{(r+1)} - (n + r)\, s_{n-1}^{(r+1)},$$

so that (5) may be written in the form

$$\begin{aligned} S_n^{(r)} = (n + r + 1)a_0(n, r)\, s_n^{(r+1)} &+ (n+r)[a_1(n, r) - a_0(n, r)]\, s_{n-1}^{(r+1)} \\ &+ (n + r - 1)[a_2(n, r) - a_1(n, r)]\, s_{n-2}^{(r+1)} + \ldots \\ &+ (r + 1)[a_n(n, r) - a_{n-1}(n, r)]\, s_0^{(r+1)}. \end{aligned} \tag{6}$$

Moreover, since

$$S_n^{(r+1)} = S_n^{(r)} + S_{n-1}^{(r)} + \ldots + S_0^{(r)}$$

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*In his Dissertation entitled "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze" (1907), Knopp establishes the following result: If a series (1) is summable for a specified value of r by Hölder's formula, it is summable for the same value of r by Cesàro's formula (*l. c.*, pp. 20, 21). By use of certain established results relative to the limiting values approached by a function defined by a power series at points *on* the circle of convergence, Knopp then remarks that the two sums *l* must be identical. The present paper may be considered to supplement these results of Knopp in two respects: (a) it establishes the equality of the last-mentioned sums without making use of extraneous theorems on power series, thus affording a new and independent proof of Hölder's original theorem (*Math. Annalen*, Vol. XX (1882), p. 535); and (b) it considers the converse theorem.

The method which we employ was suggested by the above-mentioned investigations of Knopp. A different form of proof has recently been given by Schnee (*Math. Annalen*, Vol. LXVII (1909), pp. 110–125), the latter appearing shortly after the present paper had been finished and submitted for publication. While Schneé's proof thus has priority, it is believed that the importance of the theorem, especially in its bearing upon the theory of summable divergent series, justifies the publication of a second proof. The paper has, however, been considerably condensed owing to the above circumstance.

References to the literature of the subject may be found in the above-mentioned paper of Schnee.

we obtain by use of (6) the following relation:

$$\begin{aligned} S_n^{(r+1)} = {} & (n+r+1)a_0(n,r)s_n^{(r+1)} + (n+r)\,[a_1(n,r) - a_0(n,r) \\ & + a_0(n-1,r)]\,s_{n-1}^{(r+1)} + (n+r-1)\,[a_2(n,r) - a_1(n,r) \\ & + a_1(n-1,r) - a_0(n-1,r) + a_0(n-2,r)]\,s_{n-2}^{(r+1)} + \ldots\ldots \\ & + (r+1)\,[a_n(n,r) - a_{n-1}(n,r) + a_{n-1}(n-1,r) - a_{n-2}(n-1,r) \\ & + \ldots\ldots + a_1(1,r) - a_0(1,r) + a_0(0,r)]\,s_0^{(r+1)}. \end{aligned} \tag{7}$$

From (5) and (7) it appears that the coefficients $a_0(n,r), a_1(n,r), \ldots\ldots, a_n(n,r)$ must satisfy the following system of equations:

$$\begin{aligned} a_m(n, r+1) = (n+r-m+1)\,[& a_m(n,r) - a_{m-1}(n,r) + a_{m-1}(n-1,r) \\ & - a_{m-2}(n-1,r) + \ldots\ldots - a_0(n-m+1,r) \\ & + a_0(n-m,r)]\,; \qquad (m = 0, 1, 2, 3, \ldots\ldots, n). \end{aligned} \tag{8}$$

Since $S_n^{(1)} = (n+1)s_n^{(1)}$, as results from (2) and (4), it follows that

$$a_0(n,1) = n+1,\ a_1(n,1) = a_2(n,1) = \ldots\ldots = a_n(n,1) = 0.$$

If we start with these values (corresponding to $r = 1$), we may now compute by means of (8) the values of $a_0(n,r), a_1(n,r), \ldots\ldots, a_n(n,r)$ for any given value of r.

Thus we obtain in the first place

$$a_0(n,r) = (n+r)(n+r-1)\ldots\ldots(n+1)\,; \tag{9}$$

and when $r \geqq 2 \quad (1 \leqq m \leqq n)$,*

$$\begin{aligned} a_m(n,r) = \sum_{g=1}^{r-1} (-1)^g \frac{(m-1)\,!}{(g-1)\,!\,(m-g)\,!} \sigma_g(r)(n-m+r) & \\ (n-m+r-1)\ldots\ldots(n-m+g+1), & \end{aligned} \tag{10}$$

where $\sigma_1(r), \sigma_2(r), \ldots\ldots, \sigma_{r-1}(r)$ are functions of r only determined by means of the relations

$$\left.\begin{aligned} \sigma_1(r) & = 1 + 2 + 3 + \ldots\ldots + (r-1) = \sigma_1(r-1) + (r-1), \\ \sigma_2(r) & = \sigma_1(2) + 2\sigma_1(3) + 3\sigma_1(4) + \ldots\ldots + (r-2)\sigma_1(r-1) \\ & \qquad = \sigma_2(r-1) + (r-2)\sigma_1(r-1), \\ \sigma_3(r) & = \sigma_2(3) + 2\sigma_2(4) + 3\sigma_2(5) + \ldots\ldots + (r-3)\sigma_2(r-1) \\ & \qquad = \sigma_3(r-1) + (r-3)\sigma_2(r-1) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \sigma_{r-1}(r) & = \sigma_{r-2}(r-1) = 1. \end{aligned}\right\} \tag{11}$$

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[*] In the condensation of the present paper already referred to, the actual demonstration by mathematical induction has been omitted.

Similarly, it is desirable to at once determine the set of coefficients $\alpha_0(n, r)$, $\alpha_1(n, r), \ldots, \alpha_n(n, r)$ for which the following equation is satisfied for all (positive integral) values of n and r, it being understood that $s_0^{(r)}, s_1^{(r)}, \ldots, s_n^{(r)}$ are determined by (4):

$$s_n^{(r)} = \alpha_0(n, r)\, S_n^{(r)} + \alpha_1(n, r)\, S_{n-1}^{(r)} + \ldots + \alpha_n(n, r)\, S_0^{(r)}. \tag{12}$$

Since $S_n^{(r)} = S_n^{(r+1)} - S_{n-1}^{(r+1)}$, relation (12) may be written in the form

$$\begin{aligned} s_n^{(r)} = \alpha_0(n, r)\, S_n^{(r+1)} + [\alpha_1(n, r) - \alpha_0(n, r)]\, S_{n-1}^{(r+1)} + [\alpha_2(n, r) - \alpha_1(n, r)]\, S_{n-2}^{(r+1)} \\ + \ldots + [\alpha_{n-1}(n, r) - \alpha_n(n, r)]\, S_0^{(r+1)}. \end{aligned}$$

Moreover, from (4) we have

$$s_n^{(r+1)} = \frac{1}{n+r+1}\,[s_n^{(r)} + s_{n-1}^{(r)} + \ldots + s_0^{(r)}].$$

Thus

$$\begin{aligned} s_n^{(r+1)} = \frac{1}{n+r+1} \{ & \alpha_0(n, r) S_n^{(r+1)} + [\alpha_1(n, r) - \alpha_0(n, r) + \alpha_0(n-1, r)]\, S_{n-1}^{(r+1)} \\ & + [\alpha_2(n, r) - \alpha_1(n, r) + \alpha_1(n-1, r) - \alpha_0(n-1, r) + \alpha_0(n-2, r)]\, S_{n-2}^{(r+1)} \\ & + \cdots\cdots\cdots\cdots\cdots\cdots \\ & + [\alpha_n(n, r) - \alpha_{n-1}(n, r) + \alpha_{n-1}(n-1, r) - \alpha_{n-2}(n-1, r) \\ & + \ldots + \alpha_1(1, r) - \alpha_0(1, r) + \alpha_0(0, r)]\, S_0^{(r+1)} \}. \end{aligned}$$

The coefficients $\alpha_0(n, r), \alpha_1(n, r), \ldots, \alpha_n(n, r)$ must therefore satisfy the following system of equations:

$$\begin{aligned} \alpha_m(n, r+1) = \frac{1}{n+r+1}\, [& \alpha_m(n, r) - \alpha_{m-1}(n, r) + \alpha_{m-1}(n-1, r) \\ & - \alpha_{m-2}(n-1, r) + \ldots + \alpha_0(n-m+1, r) + \alpha_0(n-m, r)]; \\ & (m = 0, 1, 2, 3, \ldots, n). \end{aligned} \tag{13}$$

Now we know in particular that $s_n^{(1)} = \dfrac{S_n^{(1)}}{n+1}$. Whence, $\alpha_0(n, 1) = \dfrac{1}{n+1}$, $\alpha_1(n, 1) = \alpha_2(n, 1) = \ldots = \alpha_n(n, 1) = 0$, and, starting with these values (corresponding to $r = 1$), we may compute by means of (13) the values of $\alpha_0(n, r)$, $\alpha_1(n, r), \ldots, \alpha_n(n, r)$.

Thus we obtain

$$\alpha_0(n, r) = \frac{1}{(n+r)(n+r-1)\ldots(n+1)}, \tag{14}$$

and when $r \geq 2$ $(1 \leq m \leq n)$,

$$\alpha_m(n, r) = \frac{1}{(n+r)(n+r-1)\ldots(n-m+1)} \sum_{g=1}^{m} \tau_g(r) P_{m-g}(n, m, r), \tag{15}$$

where $P_0(n, m, r) = 1$ and $P_g(n, m, r)$ $(1 \leqq g \leqq m)$ represents the sum of the products of the factors $(n + r - 1), (n + r - 2), \ldots, (n + r - m + 1)$ taken g at a time, and where $\tau_1(r), \tau_2(r), \ldots, \tau_{r-1}(r)$ are functions of r only determined by the following relations:

$$\left.\begin{aligned}
\tau_1(r) &= 1 + 2 + 3 + \ldots + (r-1) = \tau_1(r-1) + (r-1),\\
\tau_2(r) &= 2\tau_1(2) + 3\tau_1(3) + 4\tau_1(4) + \ldots + (r-1)\tau_1(r-1)\\
&\qquad = \tau_2(r-1) + (r-1)\tau_1(r-1),\\
\tau_3(r) &= 3\tau_2(3) + 4\tau_2(4) + 5\tau_2(5) + \ldots + (r-1)\tau_2(r-1)\\
&\qquad = \tau_3(r-1) + (r-1)\tau_2(r-1),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
\tau_{r-1}(r) &= (r-1)\tau_{r-2}(r-1) = (r-1)!\\
\tau_r(r) &= \tau_{r+1}(r) = \tau_{r+2}(r) = \ldots = \tau_n(r) = 0.
\end{aligned}\right\} \quad (16)$$

It may be observed at this point that if we place

$$T_n^{(r)} = \frac{S_n^{(r)}}{(n+r)(n+r-1)\ldots(n+1)},$$

we shall have, as a result of (12), (14) and (15),

$$s_n^{(r)} = T_n^{(r)} + \beta_1(n, r)T_{n-1}^{(r)} + \ldots + \beta_n(n, r)T_0^{(r)}, \qquad (17)$$

where

$$\beta_m(n, r) = \frac{1}{(n+r)(n+r-1)\ldots(n+r-m+1)} \sum_{g=1}^{m} \tau_g(r) P_{m-g}(n, m, r); \qquad (1 \leqq m \leqq n), \quad (18)$$

the expressions $P_g(n, m, r)$ and $\tau_g(r)$ being those indicated above.

In view of these results we may now state the original problem of § 1 as follows, at least when the expressions $s_0^{(r)}, s_1^{(r)}, \ldots, s_n^{(r)}$ are defined by (4):

(a) Given

$$S_n^{(r)} = \sum_{m=0}^{n} a_m(n, r) s_{n-m}^{(r)},$$

where $a_m(n, r)$ $(0 \leqq m \leqq n)$ is determined by (9) and (10). To show that if $\lim_{n=\infty} s_n^{(r)} = l$ then $\lim_{n=\infty} S_n^{(r)}/D_n^{(r)} = l$. This last equation may also be written in the form $\lim_{n=\infty} S_n^{(r)}/n^r = l/r!$ since

$$\frac{S_n^{(r)}}{D_n^{(r)}} = \frac{r!\,S_n^{(r)}}{(n+1)(n+2)\ldots(n+r)}.$$

(b) Given

$$s_n^{(r)} = \sum_{m=0}^{n} \beta_m(n, r) T_{n-m}^{(r)},$$

where $\beta_0(n, r) = 1$ and $\beta_m(n, r)$ $(1 \leqq m \leqq n)$ is determined by (18). To show that if $\lim_{n=\infty} T_n^{(r)} = l$ then $\lim_{n=\infty} s_n^{(r)} = r\,!\,l$.

III. *Auxiliary Theorems on Limits.*

3. In order to prove statements (a) and (b) of § 2 we shall make use of the following general theorem in the theory of limits:

(I) Let $c_1, c_2, c_3, \ldots, c_n$ be a sequence of quantities (real or complex) such that $\lim_{n=\infty} c_n = g$, and let $b_1(n), b_2(n), b_3(n), \ldots, b_n(n)$ be a set of n functions of n each of which is positive when n is positive, and such that

$$\lim_{n=\infty} \frac{b_1(n) + b_2(n) + b_3(n) + \ldots + b_m(n)}{b_1(n) + b_2(n) + b_3(n) + \ldots + b_n(n)} = 0; \quad m = constant \geqq 0.$$

Then

$$\lim_{n=\infty} \frac{b_1(n)c_1 + b_2(n)c_2 + b_3(n)c_3 + \ldots + b_n(n)c_n}{b_1(n) + b_2(n) + b_3(n) + \ldots + b_n(n)} = g.$$

In fact, we shall have under the above hypotheses $c_n = g + \varepsilon_n$, $\lim_{n=\infty} \varepsilon_n = 0$; so that, having chosen an arbitrarily small positive constant ε, we may determine a value m such that for all $n > m$ we may write the following relations, in which for brevity we place $b_1(n) = b_1$, $b_2(n) = b_2$, etc.:

$$\left|\frac{b_1c_1 + b_2c_2 + \ldots + b_nc_n}{b_1 + b_2 + \ldots + b_n} - g\right|$$

$$= \left|\frac{b_1c_1 + b_2c_2 + \ldots + b_mc_m - g(b_1 + b_2 + \ldots + b_m) + \varepsilon_{m+1}b_{m+1} + \varepsilon_{m+2}b_{m+2} + \ldots + \varepsilon_n b_n}{b_1 + b_2 + \ldots + b_n}\right|$$

$$< \left|\frac{b_1c_1 + b_2c_2 + \ldots + b_mc_m - g(b_1 + b_2 + \ldots + b_m)}{b_1 + b_2 + \ldots + b_m}\right| + \varepsilon \frac{b_{m+1} + b_{m+2} + \ldots + b_n}{b_1 + b_2 + \ldots + b_n}$$

$$< (C + g)\frac{b_1 + b_2 + \ldots + b_m}{b_1 + b_2 + \ldots + b_n} + \varepsilon\left(1 - \frac{b_1 + b_2 + \ldots + b_m}{b_1 + b_2 + \ldots + b_n}\right),$$

where C is the greatest of the quantities $|c_1|, |c_2|, \ldots, |c_m|$. Thus the truth of (I) becomes evident.

We add that in case the functions $b_1(n), b_2(n), \ldots, b_n(n)$ are not assumed to be positive, we obtain in like manner the following result:

(II) Let $c_1, c_2, c_3, \ldots, c_n$ be a sequence of quantities (real or complex) such that $\lim_{n=\infty} c_n = g$ and let $b_1(n), b_2(n), \ldots, b_n(n)$ be a set of n functions of n satisfying the following two conditions:

$$\lim_{n=\infty}\frac{|b_1(n)|+|b_2(n)|+\dots+|b_m(n)|}{b_1(n)+b_2(n)+\dots+b_n(n)}=0;\quad m=\textit{constant}\geq 0,$$

$$\frac{|b_{m+1}(n)|+|b_{m+2}(n)|+\dots+|b_n(n)|}{|b_1(n)+b_2(n)+\dots+b_n(n)|}<K;\ \begin{cases}K=\textit{constant independent of } n,\\ \quad n>n_0=\textit{constant}.\end{cases}$$

Then

$$\lim_{n=\infty}\frac{b_1(n)c_1+b_2(n)c_2+\dots+b_n(n)c_n}{b_1(n)+b_2(n)+\dots+b_n(n)}=g.$$

IV. *Proof of Theorem.*

4. We turn now to the proof of statement (a) of § 2.

From (II) it follows that if $\lim_{n=\infty} s_n^{(r)}=l$ we may write

$$\lim_{n=\infty}\frac{\sum_{m=1}^{n}a_m(n,r)s_{n-m}^{(r)}}{\sum_{m=1}^{n}a_m(n,r)}=l, \tag{19}$$

provided we can show that

$$\lim_{n=\infty}\frac{\sum_{m=n-m_1+1}^{n}|a_m(n,r)|}{\sum_{m=1}^{n}a_m(n,r)}=0;\quad m_1=\textit{constant independent of } n, \tag{20}$$

$$\frac{\sum_{m=1}^{n-m_1}|a_m(n,r)|}{\left|\sum_{m=1}^{n}a_m(n,r)\right|}<K;\ \begin{cases}K=\textit{constant independent of } n,\\ \quad n>n_0=\textit{constant}.\end{cases} \tag{21}$$

From (5) and (9) the relation (19) is seen to be equivalent to the following:

$$\lim_{n=\infty}\frac{1}{n^r}\left[S_n^{(r)}-(n+r)(n+r-1)\dots(n+1)s_n^{(r)}\right]=l\lim_{n=\infty}\sum_{m=1}^{n}a_m(n,r).$$

Thus, statement (a) will follow if we can establish (20) and (21) together with the relation

$$1+\lim_{n=\infty}\frac{1}{n^r}\sum_{m=1}^{n}a_m(n,r)=\frac{1}{r!}. \tag{22}$$

Now, from (10) we see that when $n=\infty$ the numerator of (20) becomes infinite like n^{r-2}, it being the sum of m_1 terms each of which has this same property. However, the denominator of (20) becomes infinite like n^r when $n = \infty$. To see this we note that from (10) we have

$$\sum_{m=1}^{n} a_m(n, r) = \sum_{g=1}^{r-1} \frac{(-1)^g \sigma_g(r)}{(g-1)!} \sum_{m=1}^{n} \frac{(m-1)!}{(m-g)!} (n-m+r) \\ (n-m+r-1) \ldots (n-m+g+1)$$

$$= \sum_{g=1}^{r-1} \frac{(-1)^g \sigma_g(r)}{(g-1)!} \sum_{k=1}^{n} \frac{(n-k)!}{(n-k-g+1)!} (k+r-1) \\ (k+r-2) \ldots (k+g). \quad (23)$$

Moreover, we may write

$$\frac{(n-k)!}{(n-k-g+1)!} = (n-k)(n-k-1) \ldots (n-k-g+2)$$

$$= -\sum_{p=1}^{g} \frac{(-1)^p (g-1)!}{(g-p)!\,(p-1)!} n^{g-p} \{(k+g-1)(k+g-2) \ldots (k+g-p+1) + \theta_p(k)\},$$

where $\theta_p(k)$ is of degree $p-1$ in k. The second member of (23) may therefore be put into the form

$$-\sum_{g=1}^{r-1} (-1)^g \sigma_g(r) \sum_{p=1}^{g} \frac{(-1)^p n^{g-p}}{(g-p)!\,(p-1)!} \left\{ \sum_{k=1}^{n} (k+r-1)(k+r-2) \ldots \right. \\ \left. (k+g-p+1) + \theta_p(k) \right\}$$

$$= -\sum_{g=1}^{r-1} (-1)^g \sigma_g(r) \sum_{p=1}^{g} \frac{(-1)^p n^{g-p}}{(g-p)!\,(p-1)!} \left\{ \frac{(n+r)(n+r-1) \ldots (n+g-p+1)}{r-g+p} \right. \\ \left. + \psi_p(n) \right\},$$

where $\psi_p(n)$ is of degree p in n. Thus, it not only appears that the denominator of (20) becomes infinite like n^r when $n = \infty$, but that the coefficient of n^r in the same expression is

$$\sum_{g=1}^{r-1} (-1)^g \sigma_g(r) \sum_{p=1}^{g} \frac{(-1)^{p+1}}{(g-p)!\,(p-1)!\,(r-g+p)} = \sum_{g=1}^{r-1} (-1)^g \sigma_g(r) \frac{(r-g)!}{r!}.$$

Equation (22) may therefore be replaced by the following:

$$1 + \sum_{g=1}^{r-1} (-1)^g \sigma_g(r) \frac{(r-g)!}{r!} = \frac{1}{r!}. \quad (24)$$

From the above conclusions it appears that equation (20) is satisfied; also equation (21), since the same reasoning shows directly that both the numerator and denominator of (21) become infinite like n^r when $n = \infty$. It remains then but to establish (24).

If we multiply the first member of (24) by $\frac{1}{r+1}$, using for this purpose the form

$$\frac{1}{r+1} = 1 - \frac{r}{r+1} = 1 - r\frac{r!}{(r+1)!},$$

and subsequently writing

$$\frac{(r-g)!}{(r+1)r!} = \frac{1}{(r+1)!}[(r-g+1)! - (r-g)(r-g)!],$$

we obtain

$$1 - \frac{r!}{(r+1)!}[r + \sigma_1(r)] + \sum_{g=2}^{r-1} \frac{(-1)^g (r-g+1)!}{(r+1)!}[(r-g+1)\sigma_{g-1}(r) + \sigma_g(r)] + \frac{(-1)^r}{(r+1)!},$$

which, by use of (11), becomes

$$1 + \sum_{g=1}^{r} (-1)^g \sigma_g(r+1) \frac{(r-g+1)!}{(r+1)!}.$$

But this is the expression obtained from the first member of (24) by replacing r by $r+1$, and since (24) is at once verifiable for $r=2$, it follows that it is true for $r=r$. Hence, statement (a) becomes established.

We proceed to establish in an analogous manner statement (b). Since the coefficients $\beta_m(n, r)$ $(0 \leqq m \leqq n)$ are all positive, we may evidently employ in this case the result (I) of § 3.

Thus, we know that if $\lim\limits_{n=\infty} T_n^{(r)} = l$ then

$$\lim_{n=\infty} \frac{\sum\limits_{m=1}^{n} \beta_m(n, r) T_{n-m}^{(r)}}{\sum\limits_{m=1}^{n} \beta_m(n, r)} = l, \tag{25}$$

provided that

$$\lim_{n=\infty} \frac{\sum\limits_{m=n-m_1+1}^{n} \beta_m(n, r)}{\sum\limits_{m=1}^{n} \beta_m(n, r)} = 0; \quad m_1 = \textit{constant independent of } n.$$

But equation (25) is equivalent to

$$\lim_{n=\infty} [s_n^{(r)} - T_n^{(r)}] = -1 + \lim_{n=\infty} s_n^{(r)} = l \lim_{n=\infty} \sum_{m=1}^{n} \beta_m(n, r).$$

The desired result (b) will therefore follow if we can establish (26) together with the relation

$$1 + \lim_{n=\infty} \sum_{m=1}^{n} \beta_m(n, r) = r\,! . \tag{27}$$

Moreover, relation (26) will follow at once as soon as (27) is established, since it appears from (18) that the numerator of (26) vanishes when $n = \infty$, it being the sum of m_1 terms each of which has this property. It therefore remains but to establish (27).

If we now write the expression for

$$\sum_{m=1}^{n} \beta_m(n, r)$$

as determined by (18) it is readily seen to be of the form

$$\sum_{g=1}^{r-1} f_g(n, r)\,\tau_g(r),$$

where $f_1(n, r) = \dfrac{n}{n+r}$ and $f_g(n, r)\ (2 \leqq g \leqq r-1)$ is determined by the following:

$$f_g(n, r) = \frac{1}{n+r}\,[f_{g-1}(n-1) + f_{g-1}(n-2) + \ldots\ldots + f_{g-1}(1)].$$

Thus it appears that $\lim_{n=\infty} f_g(n, r) = 1$, so that (27) may be replaced by

$$1 + \sum_{g=1}^{r-1} \tau_g(r) = r\,! .$$

This relation is now easily established, since we have

$$(1 + r)\Big[1 + \sum_{g=1}^{r-1} \tau_g(r)\Big] \equiv 1 + [\tau_1(r) + r] + [\tau_2(r) + r\tau_1(r)] + \ldots\ldots \\ + [\tau_{r-1}(r) + r\tau_{r-2}(r)] + r\tau_{r-1}(r),$$

which from (16) reduces to

$$1 + \sum_{g=1}^{r} \tau_g(r + 1).$$

The theorem of § 1 thus becomes established, provided the expressions $s_0^{(r)}$, $s_1^{(r)}$, $s_2^{(r)}$, ..., $s_n^{(r)}$ are defined by (4).

5. It remains to show that the systems (3) and (4) are equivalent in the sense in which we have just shown (2) and (4) to be equivalent. In order to avoid duplicity of notation let us suppose hereafter that the expressions $s_n^{(0)}$, $s_n^{(1)}$, $s_n^{(2)}$, ..., $s_n^{(r)}$ of (3) are called $y_n^{(0)}$, $y_n^{(1)}$, $y_n^{(2)}$, ..., $y_n^{(r)}$ respectively.

We proceed to determine n coefficients $\gamma_0(n, r), \gamma_1(n, r), \ldots, \gamma_n(n, r)$, each a function of n and r and such that for all possible values of these quantities we shall have

$$y_n^{(r)} = \gamma_0(n, r)\, s_n^{(r)} + \gamma_1(n, r)\, s_{n-1}^{(r)} + \ldots + \gamma_n(n, r)\, s_0^{(r)}. \tag{28}$$

The process by which this may be carried out closely resembles that by which the coefficients $a_0(n, r), a_1(n, r), \ldots, a_n(n, r)$ of (5) were determined, and we shall therefore merely indicate the essential steps.

From the relations

$$\left.\begin{aligned} s_n^{(r)} &= (n+1)\, s_n^{(r+1)} - n\, s_{n-1}^{(r+1)}, \\ y_n^{(r+1)} &= \frac{1}{n+r+1}\left[y_n^{(r)} + y_{n-1}^{(r)} + \ldots + y_0^{(r)}\right], \end{aligned}\right\} \tag{29}$$

in conjunction with (28), we find that

$$\begin{aligned} \gamma_m(n, r+1) = \frac{n-m+1}{n+r+1}\,[\gamma_m(n, r) &- \gamma_{m-1}(n, r) + \gamma_{m-1}(n-1, r) \\ - \gamma_{m-2}(n-1, r) + \ldots &- \gamma_0(n-m+1, r) + \gamma_0(n-m, r)]\,; \\ &(m = 0, 1, 2, 3, \ldots, n). \end{aligned} \tag{30}$$

Now, we have by inspection $\beta_0(n, 2) = \frac{n+1}{n+2}$. Thus

$$\gamma_0(n, r) = \frac{(n+1)^{r-1}}{(n+2)(n+3)\ldots(n+r)}.$$

Moreover, since we have in like manner $\gamma_1(n, 2) = 0$, while $\gamma_0(n, r)$ as just determined is such that $\gamma_0(n, r) - \gamma_0(n-1, r)$ vanishes like $1/n^2$ when $n = \infty$, it follows from (30) that when $n = \infty$ the coefficient $\gamma_1(n, r)$ vanishes like $1/n^2$. Similarly, we find that $\gamma_2(n, r)$ has the same property. More generally, it appears that when $n = \infty$ the coefficient $\gamma_m(n, r)$ $(1 \leqq m \leqq n)$ vanishes to as high an order as that of the expression

$$\frac{n-m+1}{n+r}\left(\frac{1}{n^2}\right)(1 + 2 + 3 + \ldots + m),$$

the order of which is the same as that of $\frac{m}{n^2}$. In other words,

$$\lim_{n=\infty} \frac{n^2 \gamma_m(n, r)}{m} = \textit{constant (0 incl.)}; \qquad (1 \leqq m \leqq n).$$

When $\lim\limits_{n=\infty} s_n^{(r)} = l$ we may therefore write

$$\lim_{n=\infty} \sum_{m=1}^{n} \gamma_m(n, r)\, s_{n-m}^{(r)} = 0,$$

and hence

$$\lim_{n=\infty} y_n^{(r)} = \lim_{n=\infty} \beta_0(n, r)\, s_n^{(r)} = \lim_{n=\infty} s_n^{(r)} = l.$$

In order to establish the converse of this result, we proceed to determine the coefficients $\delta_m(n, r)$ for which the following relation is satisfied:

$$s_n^{(r)} = \delta_0(n, r)\, y_n^{(r)} + \delta_1(n, r)\, y_{n-1}^{(r)} + \ldots. + \delta_n(n, r) y_0^{(r)}:$$

From the relations

$$y_n^{(r)} = (n + r + 1) y_n^{(r+1)} - (n + r) y_{n-1}^{(r+1)},$$

$$s_n^{(r+1)} = \frac{1}{n+1}\left[s_n^{(r)} + s_{n-1}^{(r)} + \ldots. + s_0^{(r)}\right],$$

we find that

$$\delta_m(n, r + 1) = \frac{n + r - m + 1}{n + 1}\,[\delta_m(n, r) - \delta_{m-1}(n, r) + \delta_{m-1}(n - 1, r)$$
$$- \ldots. + \delta_0(n - m, r]\,; \qquad (m = 0, 1, 2, \ldots., n). \quad (31)$$

Equations (31), together with the special equations $\delta_0(n, 2) = \frac{n+2}{n+1}$, $\delta_1(n, 2) = \delta_2(n, 2) = \ldots. = \delta_n(n, 2) = 0$, enable us to show as in the preceding case that if $\lim_{n=\infty} s_n^{(r)} = l$ then $\lim_{n=\infty} y_n^{(r)} = l$, thus completing the proof of the theorem of § 1.

ANN ARBOR, *August*, 1909.

Sur un Exemple de Fonction Analytique Partout Continue.

PAR D. POMPEIU.

J'ai démontré, par un exemple (*Comptes Rendus*, 28 novembre, 1904), l'existence des fonctions analytiques partout continues.

Voici cet exemple, sous sa forme la plus simple:

Considérons, dans la plan de la variable complexe u, un ensemble E: borné, parfait, partout discontinu et d'aire partout non nulle. Désignons les points de cet ensemble par ζ et par z les autres points du plan. La lettre u désignera donc les points du plan lorsqu'il n'y aura pas lieu de distinguer si ces points sont des ζ ou des z.

Cela posé, définissons une fonction ϕ par les conditions suivantes:

$$\phi(\zeta) = 1, \quad \phi(z) = 0.$$

Maintenant la fonction dont il est question est définie par l'intégrale double (au sens de M. Lebesgue)

$$F(u') = \int_{(R)} \frac{\phi}{u - u'} \cdot d\omega$$

étendue à une région (R), du plan, contenant tous les points ζ de l'ensemble E. Par $d\omega$ je désigne *l'élément d'aire* au point u.

En chaque point du plan l'intégrale double nous donne une valeur bien déterminée de F.

La fonction F, ainsi définie, est continue dans tout le plan, donc continue aussi sur l'ensemble E des points singuliers.

On sait que jusqu'à présent tous les exemples de fonctions analytiques connus présentaient cette particularité que les points *singuliers* étaient, en même temps, des points de discontinuité. La question s'est posé alors de décider si c'était là un fait essentiel tenant à la nature même des fonctions analytiques ou un pur hasard. En d'autres termes: une fonction analytique uniforme peut-elle être continue sur l'ensemble de ses points singuliers?

C'est pour répondre par l'affirmative à cette question que j'ai construit l'exemple rappelé ci-dessus.

Mais la fonction F a été considérée souvent comme non concluante, notamment parce qu'elle est identiquement nulle si l'aire de E est nulle.

Dans ces conditions M. Denjoy a cru utile de montrer, dans une Note des *Comptes Rendus* (3 mai, 1909), que la fonction F est un exemple probant. M. Denjoy précise certaines questions, pour lesquelles je m'étais rapporté à la théorie du potentiel logarithmique, et en particulier la question de la continuité de F en tous les points du plan.

Mais l'inégalité donnée, à cet effet, par M. Denjoy n'est pas assez précise. C'est pourquoi je me suis proposé de reprendre la question de la continuité de F et de rectifier, à cette occasion, une affirmation de M. Denjoy relative à la dérivée F' de F.

Je tiens, d'ailleurs, à reconnaître la rigueur et l'élégance avec lesquelles M. Denjoy met en évidence les autres propriétés de la fonction F. Ainsi il démontre que la fonction F n'est pas identiquement nulle et qu'elle admet *tous* les points ζ comme points singuliers.

§ 1. *Continuité de la fonction F.*

Pour démontrer que la fonction $F(u)$ est partout continue on doit chercher une limite supérieure de la différence

$$|F(u') - F(u'')|,$$

u' et u'' étant deux points quelconques dans le plan de la variable complexe u.

M. Denjoy trouve, aux notations près,

$$|F(u') - F(u'')| < k\,|u' - u''| \log \frac{1}{|u' - u''|}, \tag{1}$$

k étant une constante indépendante de u' et u''. Mais on voit bien que cette inégalité ne peut pas avoir lieu *quels que soient* les points u' et u''.

Je me propose de montrer que la fonction $F(u)$ satisfait, dans tout le plan, à la condition de Lipschitz

$$|F(u') - F(u'')| < k\,|u' - u''|,$$

k étant une constante indépendante de u' et u''.

On a

$$F(u') - F(u'') = (u' - u'') \int_{(E)} \frac{\phi}{(u - u')(u - u'')}\, d\omega \tag{2}$$

en désignant toujours par (R) une région du plan contenant tous les points ζ de l'ensemble E.

Tout se réduit donc à la recherche d'une limite supérieure pour le module de l'intégrale du second membre.

Pour cela je partage le plan tout entier en deux moitiés par une droite (D) perpendiculaire sur le segment $u'u''$ et passant par le milieu $\frac{1}{2}(u'+u'')$ de ce segment.

Maintenant j'observe que

$$\left|\int_{(R)}\frac{\Phi}{(u-u')(u-u'')}\,d\omega\right| < \int_{(R)}\frac{\Phi}{r'r''}\,d\omega$$

en posant

$$r' = |u-u'|,$$
$$r'' = |u-u''|.$$

Si la droite (D) partage la région (R) en deux parties* on aura

$$\int_{(R)}\frac{\Phi}{r'r''}\,d\omega = \int_{(R')} + \int_{(R'')}$$

en désignant par (R') la partie de (R) qui contient le point u' et par (R'') la partie de la région (R) qui contient le point u''.

Pour calculer

$$I' = \int_{(R')}\frac{\Phi}{r'r''}\,d\omega$$

nous mettrons l'origine des coordonnées polaires au point u'. Alors

$$d\omega = r'\,.\,d\theta'\,.\,dr'$$

et l'intégrale I' devient

$$I' = \int_{(R')}\frac{\Phi}{r''}\,dr'\,d\theta'.$$

Intégrons d'abord par rapport à r' en laissant θ' constant. On aura

$$I' = \int_0^{2\pi} d\theta' \int_0^{\rho'} \frac{\Phi}{r''}\,dr',$$

ρ' étant une certaine fonction de θ', dépendant de la forme du périmètre de (R').

Mais

$$\int_0^{\rho'}\frac{\Phi}{r''}\,dr' \leq \int_0^{\rho'}\frac{dr'}{r''},$$

---

* On peut évidemment supposer que le périmètre de la région (R) est une courbe convexe.

puisque ϕ, d'après sa définition, est *nul* en chaque point $u=z$, et égal à *un* en chaque point $u=\zeta$.

Appliquons maintenant à l'intégrale du second membre le théorème de la moyenne: il viendra

$$\int_0^{\rho'} \frac{dr'}{r''} = \frac{\rho'}{\rho''} = \alpha(\theta')$$

en désignant par ρ'' la *moyenne* des valeurs de r'' lorsque r' varie de 0 à ρ', l'angle θ' gardant la même valeur.

Mais, comme on a, dans la région (R'),

$$r' < r''$$

et seulement pour les points u de la droite (D):

$$r' = r'',$$

on conclut

$$\frac{\rho'}{\rho''} \leq 1,$$

ou encore

$$0 < \alpha(\theta') \leq 1$$

d'où

$$I' = \int_0^{2\pi} \alpha(\theta') \, . \, d\theta' \leq 2\pi.$$

Le même procédé s'applique à l'intégrale

$$I'' = \int_{(R'')} \frac{\phi}{r' r''} \, d\omega$$

et l'on trouve

$$I'' \leq 2\pi$$

d'où, pour

$$I = \int_{(R)} \frac{\phi}{(u-u')(u-u'')} \, d\omega,$$

$$|I| < 4\pi.$$

Alors l'égalité (2) nous donne

$$|F(u') - F(u'')| < 4\pi \, |u' - u''|. \tag{3}$$

C'est donc cette formule précise qui doit remplacer l'inégalité (1) de M. Denjoy.

§ 2. *La dérivée de F est bornée.*

Dans une autre Note des *Comptes Rendus* (9 août, 1909) M. Denjoy traitant des fonctions $F(u)$ partout continues, et précisément du cas où la dérivée $F'(u)$ est bornée, ajoute: "Cas dont on n'a pas jusqu'ici formé d'exemple."

M. Denjoy pense donc que la dérivée F' de la fonction F, du § précédent, n'est pas bornée. C'est, en effet, ce qui semblerait résulter de l'inégalité (1) de M. Denjoy. Mais nous venons de voir que cette inégalité peut être précisée et remplacée par l'inégalité (3) de laquelle il résulte évidemment que la dérivée F' est bornée.

C'est donc le contraire de l'affirmation de M. Denjoy qui est vrai: on n'a pas encore formé d'exemple d'une fonction $F(u)$, partout continue, ayant la dérivée *non* bornée.

§ 3. *Autres propriétés de F.*

J'ai déjà dit que M. Denjoy s'est occupé, en dehors de la continuité, d'autres propriétés de la fonction F. Je me propose de chercher une limite supérieure du module de $F(u)$, quel que soit u, dans le plan.

On a, par définition,

$$F(u') = \int_{(R)} \frac{\Phi}{u-u'}\, d\omega.$$

Mais F étant, en dehors de l'ensemble E, holomorphe et nulle à l'infini le maximum de $|F|$ correspond à un point de E. Nous sommes donc amenés à étudier les valeur de $|F|$ pour les points ζ de E.

Prenons donc un point ζ et cherchons une limite supérieure de $|F(\zeta)|$. On a

$$F(\zeta) = \int_{(R)} \frac{\Phi}{u-\zeta}\, d\omega$$

d'où

$$|F(\zeta)| < \int_{(R)} \frac{\Phi}{r}\, d\omega$$

en posant

$$|u-\zeta| = r.$$

Prenons comme région (R) un cercle, ayant le point ζ comme centre, contenant dans son intérieur tout E et contenant sur sa circonférence au moins un point de E. Ce cercle est unique et bien déterminé: soit ρ son rayon.

Mettons l'origine des coordonnées au centre ζ de (R) et prenons des coordonnées polaires. Alors

$$d\omega = r \cdot d\theta \cdot dr$$

d'où

$$\int_{(R)} \frac{\phi}{r}\, d\omega = \int_{(R)} \phi \cdot dr \cdot d\theta.$$

J'augmente cette intégrale en prenant, partout dans le cercle (R), $\phi = 1$. Donc

$$|F(\zeta)| < \int_0^{2\pi} d\theta \int_0^{\rho} dr.$$

Ou, enfin,

$$|F(\zeta)| < 2\pi\rho.$$

Mais en variant le point ζ pris comme centre de (R), la plus grande valeur que peut prendre ρ c'est δ, *diamètre* (Jordan: I, 28) de l'ensemble E.

Donc l'on a

$$|F(u)| < 2\pi\delta$$

quel que soit le point u, dans le plan.

UNIVERSITÉ DE JASSY (ROUMANIE), *Décembre*, 1909.

Symmetric Binary Forms and Involutions.—Continued.

By Arthur B. Coble.

§ 11. *Introduction.*

This paper is a continuation of two others under the same title in this Journal.* The numbering of paragraphs and theorems follows that of the earlier papers.

As a preliminary to the study of the symmetric form $H_{2,3}$ some facts and formulae which refer to the forms $H_{1,3}$ and $H_{0,3}$ are necessary.

The binary cubic

$$(a\bar{x})^3 = -[a_2^3\bar{x}_1^3 - 3a_2^2a_1\bar{x}_1^2\bar{x}_2 + 3a_2a_1^2\bar{x}_1\bar{x}_2^2 - a_1^3\bar{x}_2^3]$$

will be represented by a point X in a space S_3, and the binary cubic, $(ax)^3$, similarly expanded will be represented by a plane U if we write

$$\begin{array}{llll} X_0 = a_2^3, & X_1 = -3a_2^2a_1, & X_2 = 3a_2a_1^2, & X_3 = -a_1^3; \\ U_0 = a_1^3, & U_1 = a_1^2a_2, & U_2 = a_1a_2^2, & U_3 = a_2^3. \end{array}$$

Evidently point X and plane U determined by the same binary cubic are point and null-plane in the null-system set up by a cubic norm-curve, N_3, in S_3.

The condition that the four points, $X_{(1)}$, $X_{(2)}$, $X_{(3)}$, $X_{(4)}$, or the four planes, $U_{(1)}$, $U_{(2)}$, $U_{(3)}$, $U_{(4)}$, which represent the cubics, $(ax)^3$, $(bx)^3$, $(cx)^3$, $(dx)^3$, shall, respectively, lie in a plane, or lie on a point, is that the four cubics be linearly related. The condition that plane U given by $(ax)^3$ and point X given by $(b\bar{x})^3$ be incident is that the cubics be apolar. The formulae of transition are

$$\begin{aligned} (101) \qquad (X_{(1)}\,X_{(2)}\,X_{(3)}\,X_{(4)}) &= 9\,(ab)(ac)(ad)(bc)(bd)(cd), \\ (U_{(1)}\,U_{(2)}\,U_{(3)}\,U_{(4)}) &= (ab)(ac)(ad)(bc)(bd)(cd), \\ (UX) &= (ab)^3.\dagger \end{aligned}$$

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* Vol. XXXI, p. 183 and p. 355.

† In writing the binary form which corresponds to (UX) we shall, as a matter of convention, always put first in the apolarity condition the cubic which corresponds to U.

The symmetric form

$$H_{1,3} = (a_1 x_1)^3 (a_2 x_2)^3$$

is represented by a quadric. The point pair, $(a\bar{x})^3$, $(b\bar{x})^3$, is apolar to the quadric if $(a_1 a)^3 (a_2 b)^3 = 0$. The three values of x_2 determined by $H_{1,3}$ when x_1 is given are the parameters on N_3 of the three points in which the polar plane of the point, x_1, on N_3 as to the quadric meets N_3.

From (15) and (16) or directly from (101) we find the binary comitants of the form $H_{1,3}$ which correspond to the quaternary comitants (polarized) of the quadric. The formulae are

$$(102) \qquad \begin{aligned} a_x a_y = a'_x a'_y &= (a_1 x_1)^3 (a_2 x_2)^3 = (a'_1 x_1)^3 (a'_2 x_2)^3 = \ldots, \\ (a a' a'' u)(a a' a'' v) &= (a_1 a'_1)(a_1 a''_1)(a'_1 a''_1)(a_2 a'_2)(a_2 a''_2)(a'_2 a''_2) \\ &\qquad (a_1 \bar{x}_1)(a'_1 \bar{x}_1)(a''_1 \bar{x}_1)(a_2 \bar{x}_2)(a'_2 \bar{x}_2)(a''_2 \bar{x}_2) \\ &= (a_1 \bar{x}_1)^3 (a_2 \bar{x}_2)^3, \\ (a a' a'' a''')^2 &= (a_1 a_1)^3 (a_2 a_2)^3 \\ &= (a_1 a'_1) \ldots (a''_1 a'''_1)(a_2 a'_2) \ldots (a''_2 a'''_2). \end{aligned}$$

Also let

$$N_x N_y = (x_1 x_2)^2 \,.\, (a x_1)(a x_2)$$

be the ∞^2 quadrics which contain N_3. A member of the system is singled out by the quadratic, $(a x)^2$, which determines the two tangent lines of N_3 which also lie on the quadric.

The apolarity of the quadric $H_{1,3}$ in planes with all the quadrics containing N_3, *i. e.*,

$$(a a' a'' N)^2 = (a_1 a_2)^2 (a_1 x)(a_2 x) \equiv 0,$$

has been found in (23) to be the condition that the form $H_{1,3}$ determine an involution $I_{1,3}$. If this condition is satisfied and, at the same time, $(a a' a'' a''')^2 = 0$, then $H_{1,3}$ in planes is the square of the double point of $H_{1,3}$, and this double point must lie on N_3. The parameter of the double point is the sixfold root of $(a_1 x)^3 (a_2 x)^3 = 0$. The $I_{1,3}$ consists of an $I_{1,2}$ and a neutral point, the double point itself. In fact the quadric cone $H_{1,3}$ and the quadric cone which projects N_3 from the double point of $H_{1,3}$ are cut by any plane in apolar conics in the involution relation of § 6.

If $H_{1,3}$ is a pair of planes, the condition, $(a a' a'' N)^2 = 0$, is satisfied, but it is clear that $H_{1,3}$ will have the involution property only when the axis of the planes cuts N_3 twice. The $I_{1,3}$ is then made up of two points, each neutral, whose parameters are determined by the axis, and an $I_{1,1}$.

§ 12. *The Form $H_{2,3}$ as a Cubic Surface.*

Writing as usual

$$H_{2,3} = (a_1 x_1)^3 (a_2 x_2)^3 (a_3 x_3)^3 = (a'_1 x_1)^3 (a'_2 x_2)^3 (a'_3 x_3)^3 = \dots,$$

we recall the representations of $H_{2,3}$ as a *parametric spread* and an *apolarity spread* with respect to N_3.

If x_1, x_2, x_3 satisfy the relation, $H_{2,3} = 0$, the planes of N_3 with these parameters meet in a point X of the parametric spread which is manifestly a general cubic surface, Γ_3. If $x_1, \dots, x_5$ are an involutive set of $H_{2,3}$, the corresponding five planes of N_3 form "an inscribed 5-plane of Γ_3" (§ 3).

If x_1, x_2, x_3 satisfy the relation, $H_{2,3} = 0$, the points of N_3 with these parameters form an apolar triad of the apolarity spread which is also a general cubic surface, C_3. The point $\bar{X}$ which represents the cubic $(a\bar{x})^3 = (a'\bar{x})^3 = (a''\bar{x})^3$ lies on C_3 if $(a_1 a)^3 (a_2 a')^3 (a_3 a'')^3 = 0$. If $x_1, \dots, x_5$ are an involutive set of $H_{2,3}$, the corresponding five points of N_3 form "an antorthic 5-point or $A^5_{3,3}$ of C_3" (§ 2).

From (23) the identical vanishing of the form

$$(103) \quad (a_3 x)^3 (a'_3 x)^3 (a''_3 x)^3 (\alpha_1 \alpha_2)^2 (\alpha_1 y)(\alpha_2 y) \equiv c_0 y_2^2 - 2 c_1 y_1 y_2 + c_2 y_1^2,$$

where the symbols α are defined in (102), is the condition that $H_{2,3}$ be an $I_{2,3}$. This gives three relations, $c_k = 0$, each of the ninth degree in x, which must be identically satisfied. If C_3 has an $A^5_{3,3}$ on N_3 with parameters $x_1, \dots, x_5$, the polar quadric of any one has the other four points as an $A^4_{2,2}$, whence it is apolar to all the quadrics containing N_3 and for this particular x the form (103) vanishes independently of y. Hence the quintic $\Pi(x x_i)$, $i = 1, 2, \dots, 5$, is contained as a factor in the three equations, $c_k = 0$. If C_3 has a second $A^5_{3,3}$ on N_3 with points entirely distinct from those of the first, the equations, $c_k = 0$, of degree 9 must be satisfied by the parameters $x_1, \dots, x_{10}$ of the two $A^5_{3,3}$'s and must vanish identically. Thus $H_{2,3}$ is an $I_{2,3}$ if it has two distinct involutive sets. Again let x_1, x_2 be a neutral pair of $H_{2,3}$. Then the polar quadric of x_1 as to C_3 has a double point at x_2, and vice versa; *i. e.*, the point pair x_1, x_2 on N_3 is a pair of corresponding points on the Hessian, H, of C_3. Each polar quadric is apolar to all the quadrics on N_3, whence each equation, $c_k = 0$, has the factor $(x x_1)(x x_2)$. If then $H_{2,3}$ has five distinct neutral pairs, the equations, $c_k = 0$, must vanish identically and $H_{2,3}$ is an $I_{2,3}$. Since the general $I_{2,3}$ arises from the meets of a plane rational quintic, $R_5^{(2)}$, with lines of the plane, and since the neutral pairs of $I_{2,3}$ correspond to the six double points of $R_5^{(2)}$, the existence of five neutral pairs must imply that of a sixth. Hence:

(104) *If the form $H_{2,3}$ has two distinct involutive sets, or if it has five (and à fortiori, six) neutral pairs, the form is an $I_{2,3}$.*

This algebraic result can be applied to the parametric spread, Γ_3. If x_1, x_2 is a neutral pair of $H_{2,3}$, the planes x_1 and x_2 of N_3 must meet in a line of the surface Γ_3. Two neutral pairs of $H_{2,3}$ determine two lines of Γ_3, necessarily skew. Denote by L_i and M_i, $i = 1, 2, \ldots, 6$, the two sets of six lines of a double six of Γ_3. Then, if $H_{2,3}$ is an $I_{2,3}$, N_3 has the six skew lines L_i of a double six of Γ_3 for axes. Using the dual of the theorem that on a cubic surface there is a system of ∞^2 cubic curves which cut a set of lines L_i twice, we find, corresponding to the 72 sets of L_i on Γ_3, 72 systems of ∞^2 curves N_3 with a set of lines L_i as axes. On each N_3 there is a system of ∞^2 five-planes inscribed in Γ_3 and in an $I_{2,3}$ on N_3. These curves N_3 will be called "parametric involution curves of Γ_3."

We shall now seek to determine the $I_{2,3}$ on the involution curve N_3. Keeping in mind the connection of $I_{2,3}$ with $R_5^{(2)}$, we shall call the parameters of the double points d_i of $R_5^{(2)}$, x_{i1}, x_{i2} respectively, $i = 1, 2, \ldots, 6$; and call the parameter of the further meet of the line $\overline{d_i d_k}$ with $R_5^{(2)}$, d_{ik}. The same notation is transferred to the parameters of the planes of N_3. If a line cuts $R_5^{(2)}$ in five points, the corresponding 5-plane is inscribed in Γ_3. The inscribed 5-planes of Γ_3 on N_3 which contain the neutral pair x_{i1}, x_{i2} each contain three other planes whose parameters are determined by a pencil of binary cubics, whence the points X of Γ_3 lie on a line. Since every other neutral pair is contained in some one cubic, this line cuts every line L_k where $k \neq i$, and must be M_i. Thus the three planes of N_3 on any point of M_i together with the two planes of N_3 on L_i make up an inscribed 5-plane of Γ_3. Let $\overline{M_i L_k}$, $i \neq k$, be the meet of the lines M_i and L_k. The planes on $\overline{M_i L_k}$ are x_{k1}, x_{k2}, d_{ik} and on $\overline{M_k L_i}$ are x_{i1}, x_{i2}, d_{ik}; *i. e.*, the plane d_{ik} of N_3 contains the line $\overline{\overline{M_i L_k}\,\overline{M_k L_i}}$.

The double six $L_i M_i$ is a self-dual configuration and determines, in addition to the point spread on it, a cubic envelope Δ_3. On Δ_3, of course, the lines are double lines. The other fifteen double lines are $\overline{\overline{M_i L_k}\,\overline{M_k L_i}}$. All the planes of N_3 are planes of Δ_3; two planes are on each line L_i, one (namely d_{ik}) on each line $\overline{\overline{M_i L_k}\,\overline{M_k L_i}}$ and none on each line M_i. If therefore the planes of Δ_3 be mapped on the points of a plane Π in the well-known way — by coordinating the six lines L_i to the six fundamental points $\bar{d}_i$ of Π — the planes of N_3 are mapped on a rational quintic $\bar{R}_5^{(2)}$. This $\bar{R}_5^{(2)}$ can be only a projective modification of $R_5^{(2)}$. For the 15 lines $\bar{d}_i \bar{d}_k$ cut $\bar{R}_5^{(2)}$ in points with parameters $x_{i1}, x_{i2}, x_{k1}, x_{k2}, d_{ik}$ which are precisely the points of $R_5^{(2)}$ cut out by the line $d_i d_k$. Hence the $I_{2,3}$ of $\bar{R}_5^{(2)}$ is the same as that of $R_5^{(2)}$. Since the $I_{2,3}$ of $\bar{R}_5^{(2)}$ is cut out by the lines of Π, it will be cut out on N_3 by the ∞^2 cubic curves on Δ_3 which have the six lines M_i as axes and which have, therefore, five planes in common with N_3. Summarizing the above, we find that:

(105) *If two distinct 5-planes of N_3 are inscribed in Γ_3, or if N_3 has five (and à fortiori, six) lines L_i of Γ_3 as axes, then there are ∞^2 5-planes of N_3 inscribed in Γ_3 and lying in an $I_{2,3}$. There are 72 systems of ∞^2 involution curves N_3. The $I_{2,3}$ on N_3 is cut out by the system of involution curves whose axes are the six lines M_i which form with L_i a double six of Γ_3. One finds thus, among the ∞^5 inscribed five-planes of Γ_3, 36 systems of ∞^4, each system being determined by a double six of Γ_3.*

As a corollary there follows:

(106) *The rational invariant system of an $R_5^{(2)}$ coincides with the simultaneous rational invariant system of a cubic surface (after the adjunction of the irrationality which separates one of the sets of six lines of a double six on the surface) and a cubic curve with the six lines as axes.*

This corollary differs in geometric statement, and not in essential algebraic significance, from that derived at once by mapping the plane of $R_5^{(2)}$ on a cubic surface by means of the six double points of $R_5^{(2)}$.

§ 13. *The Apolarity Cubic Surface of the Form $I_{2,3}$.*

We have seen that if C_3 is the apolarity surface of an $I_{2,3}$, then the norm-curve N_3 can be defined to be a proper cubic space curve which cuts the Hessian, H, of C_3 in six pairs of corresponding points. Such cubic curves will be called "apolarity involution curves" of C_3. $I_{2,3}$ contains six absolute constants, while C_3 contains four at most. If, for the general $I_{2,3}$, C_3 itself is general, it must have ∞^2 involution curves. But if C_3 is necessarily subject to k conditions, it must have ∞^{2+k} involution curves. We shall prove that C_3 is general by actually finding involution curves of the general surface.

Let the Sylvester pentahedron of C_3 be $x_1, x_2, \ldots, x_5$, where $\Sigma x_i = 0$. Denote the five planes x_i by Π_i, their ten lines of intersection by Π_{ik}, and their ten points of intersection by Π_{ikl}; $i, k, l = 1, 2, \ldots, 5$. Take for the equations of C_3 and H

$$C_3 = \frac{x_1^3}{\alpha_1^2} + \frac{x_2^3}{\alpha_2^2} + \frac{x_3^3}{\alpha_3^2} + \frac{x_4^3}{\alpha_4^2} + \frac{x_5^3}{\alpha_5^2} = 0,$$

$$H = \frac{\alpha_1^2}{x_1} + \frac{\alpha_2^2}{x_2} + \frac{\alpha_3^2}{x_3} + \frac{\alpha_4^2}{x_4} + \frac{\alpha_5^2}{x_5} = 0.$$

The lines Π_{ik} are on H and the points Π_{ikl} are double points of H. The polar quadrics as to C_3 of points on Π_{ik} are cones with double points at Π_{lmn}; in particular the polar quadric of Π_{ikl} is the pair of planes $\frac{x_m}{\alpha_m} = \frac{x_n}{\alpha_n}$, $\frac{x_m}{\alpha_m} = -\frac{x_n}{\alpha_n}$, which will be denoted respectively by $\begin{pmatrix} m & n \\ + & + \end{pmatrix}$ and $\begin{pmatrix} m & n \\ + & - \end{pmatrix}$. The pencil of polar quadric cones of points on Π_{ik} have four common generators,

$$\pm \frac{x_l}{\alpha_l} = \pm \frac{x_m}{\alpha_m} = \pm \frac{x_n}{\alpha_n}.$$

These lines will be denoted by $\begin{pmatrix} l & m & n \\ \pm & \pm & \pm \end{pmatrix}$.

Thus the plane $\begin{pmatrix} i & k \\ + & + \end{pmatrix}$ contains the six lines $\begin{pmatrix} i & k & l \\ + & + & \pm \end{pmatrix}$, $\begin{pmatrix} i & k & m \\ + & + & \pm \end{pmatrix}$ and $\begin{pmatrix} i & k & n \\ + & + & \pm \end{pmatrix}$. The lines of a pair such as $\begin{pmatrix} i & k & l \\ + & + & \pm \end{pmatrix}$ meet in a point Π_{ikl}, while the lines of the pair $\begin{pmatrix} i & k & l \\ + & + & \pm \end{pmatrix}$ meet those of the pair $\begin{pmatrix} i & k & m \\ + & + & \pm \end{pmatrix}$ in the four points $\begin{pmatrix} i & k & l & m \\ + & + & \pm & \pm \end{pmatrix}$, where for example $\begin{pmatrix} 1 & 2 & 3 & 4 \\ + & - & + & - \end{pmatrix}$ denotes the point with coordinates α_1, $-\alpha_2$, α_3, $-\alpha_4$, $-(\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)$. Evidently there are 40 points $\begin{pmatrix} i & k & l & m \\ \pm & \pm & \pm & \pm \end{pmatrix}$, the 8 just written being the poles of the plane Π_n of the Sylvester pentahedron.

To obtain another property of the 40 lines $\begin{pmatrix} i & k & l \\ \pm & \pm & \pm \end{pmatrix}$ quite useful for our purpose let y be a point on H, whence $\frac{\alpha^2}{y}$ is its corresponding point and $\Sigma y_i = 0$, $\Sigma \frac{\alpha_i^2}{y_i} = 0$. The points y and $\frac{\alpha^2}{y}$ will lie on a line with Π_{ikl} if $\lambda y_j + \frac{\alpha_j^2}{y_j} = 0$, $j = i, k, l$. Hence y satisfies the relations $\frac{y_i^2}{\alpha_i^2} = \frac{y_k^2}{\alpha_k^2} = \frac{y_l^2}{\alpha_l^2}$; *i. e.*, it lies on one of the four lines $\begin{pmatrix} i & k & l \\ \pm & \pm & \pm \end{pmatrix}$;

Collecting these facts, we have:

(107) *There is a set of* 40 *lines each of which passes through a double point of H and meets H further in a pair of corresponding points. The four lines* $\begin{pmatrix} i & k & l \\ \pm & \pm & \pm \end{pmatrix}$ *meet in the point* Π_{ikl}. *Cones on these four lines are the polar quadrics of points on the line* Π_{mn}. *The lines* $\begin{pmatrix} i & k & l \\ \pm & \pm & \pm \end{pmatrix}$ *and* $\begin{pmatrix} l & m & n \\ \pm & \pm & \pm \end{pmatrix}$ *are skew; the lines* $\begin{pmatrix} i & k & l \\ \pm & + & + \end{pmatrix}$ *and* $\begin{pmatrix} k & l & m \\ + & - & \pm \end{pmatrix}$ *are skew; while the lines* $\begin{pmatrix} i & k & l \\ \pm & + & + \end{pmatrix}$ *and* $\begin{pmatrix} k & l & m \\ + & + & \pm \end{pmatrix}$ *are incident at the point* $\begin{pmatrix} i & k & l & m \\ \pm & + & + & \pm \end{pmatrix}$ *which is a pole of* Π_n *and in the plane* $\begin{pmatrix} k & l \\ + & + \end{pmatrix}$ *which is a part of the polar quadric of* Π_{imn}.

Two other groupings of points will be used. The eight poles of the plane Π_n, namely $\begin{pmatrix} i & k & l & m \\ \pm & \pm & \pm & \pm \end{pmatrix}$, divide into two sets of four denoted by t_{n+} and t_{n-} according as the number of negative signs is even or odd. It is evident from the coordinates of these points that:

(108) *The eight poles of* Π_n *divide into two tetrahedra* t_{n+} *and* t_{n-} *which with the tetrahedron* $\tau_n = \Pi_i \Pi_k \Pi_l \Pi_m$ *form a desmic system. The* 16 *lines of perspection of the system are* $\begin{pmatrix} i & k & l \\ \pm & \pm & \pm \end{pmatrix}$.

We consider now the five points $\begin{pmatrix} i & k & l & m \\ + & + & + & + \end{pmatrix}$, which we denote collectively by $(+ + + + +)$ and singly by $p_1, p_2, \ldots, p_5$. The five points are poles of the five planes Π_i. The ten lines which join them are on the ten points Π_{ik}. The ten planes which join them are the planes $\begin{pmatrix} i & k \\ + & + \end{pmatrix}$. The mixed polar planes of the points two at a time are the planes $\begin{pmatrix} i & k \\ + & - \end{pmatrix}$. We find in all 16 such sets of five points corresponding to the 16 symbols $(\pm \pm \pm \pm \pm)$. A fairly inclusive statement of the incidence relations in our configuration is this:

(109) *The* 40 *poles of the five Sylvester planes can be divided in* 16 *ways into sets of five points such that each set of five points and the ten planes on them, together with the Sylvester five planes and the ten points on them, form a Desargues configuration* $(15_6, 20_3)$. *In each configuration ten lines are determined by the*

Sylvester planes; the other ten lines are found among the 40 $\begin{pmatrix} i & k & l \\ \pm & \pm & \pm \end{pmatrix}$. *Each of the sixteen configurations is self-polar as to a quadric of the type*

$$\frac{x_1^2}{\pm\alpha_1}+\frac{x_2^2}{\pm\alpha_2}+\dots+\frac{x_5^2}{\pm\alpha_5}.$$

Writing C_3 symbolically as $a_x^3=b_x^3=\dots$, we have the plane equation of the polar quadric of a point x, $a_x b_x c_x (abcu)^2=0$. If x and y are corresponding points on H, then

$$a_x b_x c_x (abcu)^2=\rho\,.\,u_y^2$$

and

$$a_y b_y c_y (abcu)^2=\sigma\,.\,u_x^2.$$

Let $\alpha=\alpha_x^2$ be any quadric; it determines an "associated cubic surface," $C_\alpha=a_x b_x c_x (abc\alpha)^2$, the locus of points x whose polar quadrics are apolar to α. C_α contains the ten points Π_{ikl}, and as α changes C_α runs through the linear system of cubics on these ten points. C_α meets H in a 12-ic curve, the locus of points which correspond to the points of the octavic curve in which α meets H. Let x be a point on H, α, and C_α, and let y be its corresponding point. Then y lies on H, C_α, and α; *i. e.*, the sextic curve in which α and C_α meet cuts H in twelve pairs of corresponding points.

(110) *There is a system of* ∞^9 *sextic curves, the meets of quadrics* α *with their associated cubics* C_α *which cut the Hessian of a cubic surface in twelve pairs of corresponding points.*

If α contains a point Π_{ikl}, C_α contains the line Π_{mn}. If α contains the four points τ_i, C_α contains the four lines Π_{ik}. Hence the plane Π_i which meets C_α in these four lines must be a part of C_α. The other part of C_α must be a quadric which also is on the vertices τ_i. Thus each quadric on τ_i determines another of this same kind. Let us consider this correspondence for the case $i=5$. A quadric on τ_5 is $\Sigma\, b_{ik} x_i x_k=0$, where $i, k=1, 2, 3, 4$ and $i\neq k$. The polar quadric of x in planes is $\sum\limits_{l,m,n=1}^{5}\dfrac{x_l x_m x_n}{\alpha_l^2\alpha_m^2\alpha_n^2}\Pi_{lmn}^2$. This is apolar to $\Sigma\, b_{ik}x_i x_k$ if $\sum\limits_{i,k=1}^{4}\dfrac{b_{ik}}{\alpha_l^2\alpha_m^2\alpha_n^2}x_l x_m x_n=0$; *i. e.*, if $\dfrac{x_5}{\alpha_5^2}\sum\limits_{i,k=1}^{4} b_{ik}\alpha_i^2\alpha_k^2 x_l x_m=0$. Hence the relation between $\Sigma\, b_{ik}x_i x_k=0$ and its associated quadrics $\Sigma\, b'_{ik}x_i x_k=0$ is given by

$$b'_{12}=\alpha_3^2\alpha_4^2 b_{34},\quad b'_{13}=\alpha_2^2\alpha_4^2 b_{24},\quad b'_{14}=\alpha_2^2\alpha_3^2 b_{23},$$
$$b'_{34}=\alpha_1^2\alpha_2^2 b_{12},\quad b'_{24}=\alpha_1^2\alpha_3^2 b_{13},\quad b'_{23}=\alpha_1^2\alpha_4^2 b_{14}.$$

Evidently this relation is involutory and the two nets of quadrics

$$\lambda\left(\frac{x_1x_2}{\alpha_1\alpha_2}-\frac{x_3x_4}{\alpha_3\alpha_4}\right)+\mu\left(\frac{x_1x_3}{\alpha_1\alpha_3}-\frac{x_4x_2}{\alpha_4\alpha_2}\right)+\nu\left(\frac{x_1x_4}{\alpha_1\alpha_4}-\frac{x_2x_3}{\alpha_2\alpha_3}\right)=0,$$

$$\rho\left(\frac{x_1x_2}{\alpha_1\alpha_2}+\frac{x_3x_4}{\alpha_3\alpha_4}\right)+\sigma\left(\frac{x_1x_3}{\alpha_1\alpha_3}+\frac{x_4x_2}{\alpha_4\alpha_2}\right)+\tau\left(\frac{x_1x_4}{\alpha_1\alpha_4}+\frac{x_2x_3}{\alpha_2\alpha_3}\right)=0$$

are each self-associated. Denote these nets by Q_{5+} and Q_{5-} respectively. The net Q_{5+} is on the eight points τ_5 and t_{5+}; the net Q_{5-} is on the eight points τ_5 and t_{5-}. On account of the involutory relation each quadric of the one net meets each quadric of the other in a quartic curve which is on the four points τ_5 and which meets H further in four pairs of corresponding points. Since each quadric in a net is associated to itself, it meets H in an octavic curve which has double points at the vertices of τ_5 and which contains ∞^1 pairs of corresponding points. Hence:

(111) *The ten nets of quadrics $Q_{i\pm}$ on the eight points of τ_i and $t_{i\pm}$ meet H in ten systems of ∞^2 self-corresponding octavic curves. The meets of the nets Q_{i+} and Q_{i-} determine five systems of ∞^4 quartic curves on the vertices of τ_i which meet H again in four pairs of corresponding points. The meets of the nets $Q_{i\pm}$ and $Q_{k\pm}$ determine forty systems of ∞^4 quartic curves which pass through Π_{lmn} and meet H again in seven pairs of corresponding points.*

To locate involution cubic curves in one of the forty systems just mentioned let us take, to fix ideas, the nets Q_{4-} and Q_{5-}. Their quartic curves of intersection pass through Π_{123}. Since, as we shall see, an involution cubic curve can not in general pass through a double point of H, if one such quartic curve is to degenerate into a line and a cubic curve, the line must pass through Π_{123} and meet H again in a pair of corresponding points. It must therefore be one of the four lines $\begin{pmatrix}1&2&3\\ \pm&\pm&\pm\end{pmatrix}$, and we shall take it to be $\begin{pmatrix}1&2&3\\ +&+&+\end{pmatrix}$. The net Q_{4-} is on the base-points τ_4 and t_{4-}. Since these are desmic, the cones of the net are found in four pencils with double points at t_{4+}. Of the vertices of t_{4+}, a single one, namely $p_4=\alpha_1, \alpha_2, \alpha_3, \backslash, \alpha_5$,* is found on the chosen line $\begin{pmatrix}1&2&3\\ +&+&+\end{pmatrix}$. The pencil of cones with vertex at p_4 and on τ_4 and t_{4-} has the base-generators $\begin{pmatrix}1&2&3\\ +&+&+\end{pmatrix}$, $\begin{pmatrix}2&3&5\\ +&+&+\end{pmatrix}$, $\begin{pmatrix}1&3&5\\ +&+&+\end{pmatrix}$, $\begin{pmatrix}1&2&5\\ +&+&+\end{pmatrix}$. Similarly, in the net Q_{5-}

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* Here and elsewhere one of the five coordinates, the negative of the sum of the other four, is indicated by the stroke \\.

there is a single pencil of cones with vertex at a point p_5 on $\begin{pmatrix} 1 & 2 & 3 \\ + & + & + \end{pmatrix}$, and this has the base-generators $\begin{pmatrix} 1 & 2 & 3 \\ + & + & + \end{pmatrix}$, $\begin{pmatrix} 2 & 3 & 5 \\ + & + & + \end{pmatrix}$, $\begin{pmatrix} 1 & 3 & 5 \\ + & + & + \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 5 \\ + & + & + \end{pmatrix}$. According to (107) the common generator of the two pencils is met by each of the other three base-generators and each one of these three meets a definite one of the three in the other pencil in a point p_i, $i = 1, 2, 3$. Hence every cone of the one pencil cuts every cone of the other in the line $\begin{pmatrix} 1 & 2 & 3 \\ + & + & + \end{pmatrix}$ and a cubic curve on the five points p_1, p_2, p_3, p_4, p_5. Moreover every such cubic curve lies on a cone of each pencil. Making use of (109) we obtain the desired result:

(112) *There are* 16 *systems of* ∞^2 *apolarity involution curves of a general cubic surface* C_3. *Each system is the totality of proper cubic curves on one of the* 16 *sets of five points* $(\pm \pm \pm \pm \pm)$.

We still have to show that, for the general $I_{2,3}$, the involution curve N_3 is found in one of the above 16 systems. Note first that N_3 can not meet a line Π_{ik}. For if it meets Π_{ik} at a point, it must contain the correspondent of the point, Π_{lmn}; in which case it could not meet H in six distinct pairs of corresponding points. Denote by C_{ikl} the chord of N_3 from the point Π_{ikl}.

Since the polar quadric of any point on N_3 is apolar to all the point quadrics on N_3, the cubic surface C_α contains N_3 if the quadric α contains N_3. Let β be a quadric on N_3. Then β and C_β meet in a sextic curve made up of N_3 and another cubic curve N_β which cuts N_3 in five points and H in six pairs of corresponding points. Hence N_β is also an involution cubic and we find from the net β, ∞^2 involution curves N_β which will be called "incident" curves of N_3.

Let β_5 be the particular quadric on N_3 and the two points Π_{123} and Π_{124}. Then C_{β_5} contains N_3 and the lines Π_{45} and Π_{35}. The plane Π_5 meets C_{β_5} in these two lines and a further line which must contain the point Π_{125} and the three points where N_3 meets Π_5. Since this is impossible, Π_5 is part of C_{β_5} and β_5 contains all four points τ_5. C_{β_5} contains N_3 and the same four points, whence C_{β_5} is $\Pi_5 . \beta_5$, or β_5 is a self-associated quadric (see p. 341), which we shall suppose to be in the net Q_{5-}. Similarly, there is a self-associated quadric β_4 on N_3 and the points τ_4, which we shall suppose to be in the net Q_{4-}.

The pencil of quadrics $\beta_4 + \lambda \beta_5$ contains N_3 and C_{123}; the associated pencil $C_{\beta_4} + \lambda C_{\beta_5}$ contains N_3, C_{123}, and Π_{45}; it is in fact $\frac{y_4}{\alpha_4} \beta_4 + \lambda \frac{y_5}{\alpha_5} \beta_5$. Each quadric of the pencil meets its associate in N_3, C_{123}, and a conic. The locus of these conics is evidently the plane $\frac{y_4}{\alpha_4} - \frac{y_5}{\alpha_5} = 0$ or $\begin{pmatrix} 4 & 5 \\ + & + \end{pmatrix}$. The conics form a pencil on the points q_1, q_2, q_3, where $\begin{pmatrix} 4 & 5 \\ + & + \end{pmatrix}$ meets N_3, and the point q where C_{123} meets $\begin{pmatrix} 4 & 5 \\ + & + \end{pmatrix}$. But each conic of the pencil together with C_{123} is a degenerate "incident" involution curve N_3 and must cut H in six pairs of corresponding points. From the symmetry of the case it is necessary that these pairs be distributed as follows: three on the conic, two for the meets of the conic with Π_{45} and the point Π_{123} on C_{123}, and therefore one further pair on C_{123}. This requires however that C_{123} be one of the four lines $\begin{pmatrix} 1 & 2 & 3 \\ \pm & \pm & \pm \end{pmatrix}$; let it be $\begin{pmatrix} 1 & 2 & 3 \\ + & + & + \end{pmatrix}$. But one verifies at once that if β_5 is a quadric in the net Q_{5-} and contains the line $\begin{pmatrix} 1 & 2 & 3 \\ + & + & + \end{pmatrix}$, it must be a cone with vertex at p_5. Similarly, β_4 is a cone of the net Q_{4-} with vertex at p_4. These two cones meet in the given N_3 and they are also precisely the cones used to determine one of the 16 systems of involution curves already found; *i. e.*, the general involution curve is found in one such system. The two possible choices of β_5 in a net $Q_{5\pm}$ and β_4 in a net $Q_{4\pm}$ and the four possible choices of the line $\begin{pmatrix} 1 & 2 & 3 \\ \pm & \pm & \pm \end{pmatrix}$ give rise to the 16 systems.

(113) Incidentally we have shown that:

In addition to the ten singular planes of the Cayleyan congruence which contain a point Π_{ikl} *and line* Π_{lm}, *the lines of the congruence being on the point, there is a set of twenty singular planes* $\begin{pmatrix} i & k \\ \pm & \pm \end{pmatrix}$. *Such a plane cuts the Hessian in the line* Π_{ik} *and a cubic curve which contains* ∞^1 *pairs of corresponding points whose joins envelope a general cubic.*

We remarked earlier that if the apolarity and parametric spreads are of the same dimensions they are covariants of each other and *a common involution norm-curve.* In our present case the connection between Γ_3 and C_3 can be readily exhibited by means of the null-system or correlation set up by N_3. The planes x_1, x_2, x_3 of N_3 which meet in a point X of Γ_3 are transformed by this correlation into points x_1, x_2, x_3 on N_3 which form an apolar triad of C_3. Hence Γ_3 is transformed into a cubic envelope E_3 whose planes cut N_3 in apolar triads of C_3. The six lines L_i of Γ_3 become the six lines L_i', joining the pairs of corresponding points in which N_3 meets H. Hence these six lines of the Cayleyan congruence belong to a double six. The six lines M_i of Γ_3 become the cross-lines, M_i', of L_i'. The cubic envelope Δ_3 on the double six $L_i M_i$ becomes a cubic surface D_3 which has the double six $L_i' M_i'$. Making use of (105), we have the theorem:

(114) *An involution curve N_3 of C_3 cuts H in six pairs of corresponding points. The six lines of the Cayleyan congruence so determined are skew lines L_i' of a double six. The cubic surface D_3 on N_3, on the six lines L_i', and on their cross-lines M_i', contains ∞^2 cubic curves with the chords M_i'. These curves cut out on N_3 the given $I_{2,3}$.*

§ 14. *The Apolarity Spread of the Rational Plane Curve of Order n.*

Some of the foregoing results concerning the apolarity cubic curve in S_2 of an $R_4^{(2)}$ and concerning the apolarity cubic surface in S_3 of an $R_5^{(2)}$ can be readily generalized. The $R_n^{(2)}$ establishes its $I_{2,n-2}$, the involution cut out by the lines of the plane. This we express by means of a symmetric form,

$$H_{2,n-2} = (a_1 x_1)^{n-2} (a_2 x_2)^{n-2} (a_3 x_3)^{n-2}.$$

The apolarity spread of the form is a cubic spread, $C_3^{(n-2)}$, in S_{n-2} referred to a norm-curve, N_{n-2}. The condition that $H_{2,n-2}$ be an $I_{2,n-2}$ is the identical vanishing of a form, F, of order $(n-2)^2$ in x and symmetric in x_1 and x_2 of order $(n-4)$ in each. The argument of § 12 for the case $n=5$ can be extended. If $H_{2,n-2}$ has an involutive set, $y_1, y_2, \ldots, y_n$, then F contains the factor, $(xy_1) \ldots (xy_n)$; or if $H_{2,n-2}$ has a neutral pair, y_1, y_2, then F contains the factor $(xy_1)(xy_2)$. If the number of these distinct factors be greater than $(n-2)^2$, F must vanish identically.

(115) *The form $H_{2,n-2}$ is an $I_{2,n-2}$ if it has an integer number greater than $\frac{(n-2)^2}{n}$ of distinct involutive sets; or if it has an integer number greater than $\frac{(n-2)^2}{2}$ $\left(\text{and à fortiori equal to } \frac{(n-1)(n-2)}{2}\right)$ of distinct neutral pairs.*

A geometric statement of the theorem is this:

(116) *If a norm-curve in S_{n-3} contains more than $\frac{(n-2)^2}{n}$ distinct antorthic sets $A_{3,3}^n$ of a cubic spread $C_3^{(n-2)}$, or if it cuts the Hessian of $C_3^{(n-2)}$ in more than $\frac{(n-2)^2}{2}$ pairs of corresponding points, then it contains ∞^2 $A_{3,3}^n$'s of $C_3^{(n-2)}$ which lie in an $I_{2,n-2}$.*

We are concerned mainly with the case of an $I_{2,n-2}$ and the $R_n^{(2)}$ from which it originates. $C_3^{(n-2)}$ cuts N_{n-2} in the $3(n-2)$ points whose parameters give the flexes of $R_n^{(2)}$. Caporali has shown for the cases $n = 4, 5$ that if $n-1$ points of an $A_{3,3}^n$ of $C_3^{(n-2)}$ lie on $C_3^{(n-2)}$, the n-th point also lies on $C_3^{(n-2)}$. His method of proof applies equally well to the general case and we translate the result thus:

(117) *If $(n-1)$ flexes of a rational plane curve of order n lie on a line, an n-th flex also lies on the line.*

We have seen that $C_3^{(2)}$ and $C_3^{(3)}$ were general cubic spreads. $C_3^{(n-2)}$ has $\frac{1}{6}(n^3 - 6n^2 + 11n - 6)$ absolute constants, while the $R_n^{(2)}$ has $3(n-3)$ absolute constants. Certainly, then, when $n > 6$, the apolarity spread of an $I_{2,n-2}$ is a *special* cubic spread. It would be interesting to know more of the conditions on the spread and of the location of the involution curves than is indicated in (116). These questions are answered in this paragraph for only two kinds of rational curves.

Take first the $R_n^{(2)}$ with an $(n-1)$-fold point. The $I_{2,n-2}$ has a set of parameters, $\lambda_1, \ldots, \lambda_{n-1}$, such that any two of the set are a neutral pair. There is on N_{n-2} a set of $n-1$ points such that any two are apolar to the cubic $C_3^{(n-2)}$. Using these as an $(n-1)$-point of reference, $C_3^{(n-2)}$ becomes

$$a_1 x_1^3 + \cdots + a_{n-1} x_{n-1}^3,$$

whence:

(118) *If $C_3^{(n-2)}$ can be expressed as a sum of $(n-1)$ cubes, any one of the $\infty^{2(n-3)}$ proper norm-curves N_{n-2} on the points of its orthic $(n-1)$-edron is an involution curve. The $A_{3,3}^n$'s of C_3^{n-2} on such a curve lie in the $I_{2,n-2}$ of an $R_n^{(2)}$ with an $(n-1)$-fold point.*

To obtain the second kind of $R_n^{(2)}$ considered here we assume that $C_3^{(n-2)}$ has an orthic n-edron, $\Pi_1, \Pi_2, \ldots, \Pi_n$, and then prove that it has involution curves. Using the n-edron as a basis of reference, we have

$$C_3^{(n-2)} = \sum_{i=1}^{n} \frac{x_i^3}{\alpha_i^2}, \text{ where } \sum_{i=1}^{n} x_i = 0.$$

The Hessian of order $n-1$ is

$$H = \sum_1^n \frac{\alpha_i^2}{x}.$$

If both $\Sigma\, y_i = 0$ and $\Sigma \frac{\alpha_i^2}{y_i} = 0$, the points y_i and $\frac{\alpha_i^2}{y_i}$ are corresponding points on H. H has simple points on the spaces Π_{ik}, double points on the spaces

Π_{ikl}, etc., the notation here being similar to that of the preceding paragraph. The configuration of poles of the spaces Π_i obtained there exists for all values of n greater than two. In particular from the n sets of 2^{n-2} poles one can in 2^{n-1} ways pick out n poles like the following:

$$\begin{array}{llllll}
1/\alpha_1\, p_1 = \alpha_1 - \alpha, & \alpha_2, & \alpha_3, & \ldots, & \alpha_n, \\
1/\alpha_2\, p_2 = \alpha_1, & \alpha_2 - \alpha, & \alpha_3, & \ldots, & \alpha_n, \\
1/\alpha_3\, p_3 = \alpha_1, & \alpha_2, & \alpha_3 - \alpha, & \ldots, & \alpha_n, \\
\multicolumn{5}{l}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,} \\
1/\alpha_n\, p_n = \alpha_1, & \alpha_2, & \alpha_3, & \ldots, & \alpha_n - \alpha,
\end{array}$$

where $\alpha = \sum_i \alpha_i$ and $\sum_i p_i \equiv 0$.

The point p_i is a pole of Π_i and the line $\overline{p_i p_k}$ passes through the $(n-3)$-fold point $\Pi_{l,m,n,\ldots}$ of the Hessian.

A norm-curve on the n points p_i and the point y is unique. Its parametric equation is:

$$\begin{array}{lll}
(119) & 1/\alpha_1\, x_1 = & \alpha_2\, \overline{12}\, t_{12} + \alpha_3\, 13\, t_{13} + \ldots + \alpha_n\, \overline{1n}\, t_{1n}, \\
& 1/\alpha_2\, x_2 = & \alpha_1\, \overline{21}\, t_{21} + \alpha_3\, \overline{23}\, t_{23} + \ldots + \alpha_n\, \overline{2n}\, t_{2n}, \\
& 1/\alpha_3\, x_3 = & \alpha_1\, \overline{31}\, t_{31} + \alpha_2\, \overline{32}\, t_{32} + \ldots + \alpha_n\, \overline{3n}\, t_{3n}, \\
& \multicolumn{2}{l}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,} \\
& 1/\alpha_n\, x_n = & \alpha_1\, \overline{n1}\, t_{n1} + \alpha_2\, \overline{n2}\, t_{n2} + \alpha_3\, \overline{n3}\, t_{n3} + \ldots,
\end{array}$$

where $\overline{ik} = \dfrac{y_i}{\alpha_i} - \dfrac{y_k}{\alpha_k}$ and $t_{ik} = \dfrac{\prod_1^n \left(t - \dfrac{y_l}{\alpha_l}\right)}{\left(t - \dfrac{y_i}{\alpha_i}\right)\left(t - \dfrac{y_k}{\alpha_k}\right)}$.

The parameter of p_i is $\dfrac{y_i}{\alpha_i}$; the parameter of y is ∞. When $t = 0$, we have the point

$$1/\alpha_i\, x_i^{(0)} = \left[\left(\frac{\alpha_1^2}{y_1} + \ldots\right) - \left(\frac{\alpha_1 \alpha_i}{y_i} + \ldots\right)\right] \prod_1^n \left(\frac{y_k}{\alpha_k}\right),$$

whence

$$x_i^{(0)} = \left[\alpha_i \left(\sum_1^n \frac{\alpha_k^2}{y_k}\right) - \frac{\alpha_i^2}{y_i}\left(\sum_1^n \alpha_k\right)\right] \prod_1^n \left(\frac{y_k}{\alpha_k}\right).$$

Since y can be any point on N_{n-2}, let it be on H, whence

$$x_i^{(0)} = -\left(\sum_1^n \alpha_k\right) \prod_1^n \left(\frac{y_k}{\alpha_k}\right) \cdot \frac{\alpha_i^2}{y_i};$$

i. e., $x_i^{(0)}$ is the corresponding point of y. Hence N_{n-2} cuts H in pairs of corresponding points and is an involution curve.

(120) *If a cubic spread $C_3^{(n-2)}$ in S_{n-2} can be expressed as a sum of n cubes, one can pick from the $2^{n-2}\,.\,n$ poles of its n orthic S_{n-3}'s in 2^{n-1} ways an n-point with the following property: the n points and the $\frac{1}{2}\,n\,(n-1)$ S_{n-3}'s on them together with the n orthic S_{n-3}'s and the $\frac{1}{2}\,n\,(n-1)$ points on them form a Desargues configuration (a section of the complete $(n+1)$-point in S_{n-1}). Every proper norm-curve N_{n-2} on such an n-point cuts the Hessian of $C_3^{(n-2)}$ in $\frac{1}{2}\,(n-1)\,(n-2)$ pairs of corresponding points and contains $\infty^2\,A_3^n$'s of $C_3^{(n-2)}$ which lie in the $I_{2,\,n-3}$ of a rational plane curve $R_n^{(2)}$ which is subject to $(n-5)$ conditions.*

For $C_3^{(n-2)}$ has only $n-1$ absolute constants and only $n-3$ are added in the choice of an N_{n-2}.

With each of the 2^{n-1} systems of involution curves there is associated a quadric with regard to which the corresponding Desargues configuration is self-polar. For the particular n-point, p_i, this quadric is

$$Q \equiv \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \ldots\ldots + \frac{x_n^2}{a_n}.$$

Let Q_x be the polar quadric of a point x as to $C_3^{(n-2)}$. Then

$$Q_{p_1} = \frac{a_1 - a}{a_1}\,\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \ldots\ldots + \frac{x_n^2}{a_n};$$

i. e., Q and Q_{p_1} touch along the space Π_1, which space has the same pole, p_1, as to both quadrics. Thus p_1 and any self-polar $(n-2)$-point of Q or Q_{p_1} in Π_1 is a self-polar $(n-1)$-point of Q and Q_{p_1}. But it is known that all the ∞^{n-3} norm-curves in our system cut Π_1 in self-polar $(n-2)$-points of Q. Thus p_1 and the $n-2$ points in which Π_1 cuts N_{n-2} constitute a self-polar $(n-1)$-point of Q on N_{n-2}. The existence of one such set implies that of ∞^1 sets which lie in an $I_{1,\,n-2}$, and Q is apolar to all the point quadrics on N_{n-2}. In this $I_{1,\,n-2}$ each of the n sets of points (p_i and the points on Π_i) is also a self-polar $(n-1)$-point of of a polar quadric Q_{p_i}, hence with the point p_i forms an $A_{3,\,3}^n$ of $C_3^{(n-2)}$. A similar fact is true of the polar quadric of any point on N_{n-2}, and we shall verify the theorem:

(121) *If $Q = \Sigma \dfrac{x_i^2}{a_i}$ is the quadric associated with the system of ∞^{n-3} involution curves in which the given N_{n-2} lies, then the polar quadric of a point τ on N_{n-2}*

has in common with Q *a self-polar* $(n-1)$*-point whose* $(n-1)$ *vertices* t *are on* N_{n-2}. *The* $(1, n-1)$ *relation between* τ *and* t *is*

$$\alpha \left\{ \prod_1^n \left(t - \frac{y_k}{\alpha_k} \right) \right\}_\tau - \frac{1}{n} (\tau t) \left\{ \frac{x_1}{\alpha_1} + \frac{x_2}{\alpha_2} + \dots + \frac{x_n}{\alpha_n} \right\} = 0,$$

where the subscript of the brace indicates the first polar of τ, *and where* $\frac{x_i}{\alpha_i}$ *is the* $(n-2)$*-ic in* t *of* (119).

The point $t = \tau$ *and the* $n-1$ *points* t *determined by* τ *together constitute an* $A^n_{3,3}$ *of* C_3^{n-2}, *whence by varying* τ *in the* $(2, n)$ *relation*

$$(\tau t) \alpha \left\{ \prod_1^n \left(t - \frac{y_k}{\alpha_k} \right) \right\}_\tau - \frac{1}{n} (\tau t)^2 \left\{ \Sigma \frac{x_k}{\alpha_k} \right\} = 0,$$

a singly infinite quadratic system of $A^n_{3,3}$*'s is obtained.*

Note first that in (119) $\frac{x_1}{\alpha_1} - \frac{x_i}{\alpha_i} = \alpha\,(1i)\,t_{1i}$ and the $(1, n-1)$ relation can be written

$$\alpha \left\{ \prod_1^n \left(t - \frac{y_k}{\alpha_k} \right) \right\}_\tau - \frac{1}{n} (\tau t) \left\{ n \frac{x_1}{\alpha_1} - \alpha \left[\overline{12}\, t_{12} + \dots + \overline{1n}\, t_{1n} \right] \right\}.$$

Let τ take the value $\frac{y_1}{\alpha_1}$. The polarized term becomes

$$\frac{1}{n} \left(t - \frac{y_1}{\alpha_1} \right) \left[\overline{12}\, t_{12} + \dots + \overline{1n}\, t_{1n} \right],$$

which is cancelled by part of the second term, whence the result is $\left(t - \frac{y_1}{\alpha_1} \right) \frac{x_1}{\alpha_1}$. Similarly for $\tau = \frac{y_i}{\alpha_i}$, the $(n-1)$-ic in t becomes $\left(t - \frac{y_i}{\alpha_i} \right) \frac{x_i}{\alpha_i}$. We have identified these $(n-1)$-ics as self-polar $(n-1)$-points of Q and two alone are sufficient to define the pencil. We have also identified the n-ics $\left(t - \frac{y_i}{\alpha_i} \right)^2 \frac{x_i}{\alpha_i}$ as $A^n_{3,3}$'s of $C_3^{(n-2)}$. The ∞^2 $A^n_{3,3}$'s lie in a linear system which must contain the ∞^1 given by varying τ in the $(2, n)$ relation, since they have n known members in common. We shall recur to the meaning of this $(1, n-1)$ relation with regard to the underlying $R_n^{(2)}$ in the next paragraph.

It is evident from the analysis that our entire apparatus is determined by the choice of the n-space Π_i, of the quadric Q for which the n-space is orthic, and of the N_{n-2} on the orthic n-point p_i of Q which is the polar of the given n-space Π_i as to Q. Since N_{n-2} contains one orthic n-point of Q, it is known to contain ∞^3 which lie in an $I_{3,\,n-3}$. These are the ∞^3 canonizant sets of the $2(n-2)$-ic determined on N_{n-2} by the S_{n-3}'s common to Q and N_{n-2}. In this $I_{3,\,n-3}$ all the sets of our given $I_{2,\,n-2}$ are found. To prove this, we have merely to show that if x and y are points on N_{n-2} their polar space as to $C_3^{(n-2)}$ has a pole as to Q which is on the line $\overline{xy}$. The condition that x and y be on N_{n-2} is the vanishing of the matrix

$$\left\Vert \begin{matrix} \dfrac{x_1}{\alpha_1} & \dfrac{x_2}{\alpha_2} & \cdots\cdot & \dfrac{x_n}{\alpha_n} \\ \dfrac{x_1 y_1}{\alpha_1^2} & \dfrac{x_2 y_2}{\alpha_2^2} & \cdots\cdot & \dfrac{x_n y_n}{\alpha_n^2} \\ \dfrac{y_1}{\alpha_1} & \dfrac{y_2}{\alpha_2} & \cdots\cdot & \dfrac{y_n}{\alpha_n} \\ 1 & 1 & \cdots\cdot & 1 \end{matrix} \right\Vert .$$

Each determinant of this matrix is, for given y on N_{n-2}, a quadric on N_{n-2}; *e. g.*, the first determinant is

$$\begin{aligned} &\overline{12}\,\overline{34}\left(\frac{x_3 x_4}{\alpha_3 \alpha_4}+\frac{x_1 x_2}{\alpha_1 \alpha_2}\right)+\overline{13}\,\overline{42}\left(\frac{x_4 x_2}{\alpha_4 \alpha_2}+\frac{x_1 x_3}{\alpha_1 \alpha_3}\right)+\overline{14}\,\overline{23}\left(\frac{x_2 x_3}{\alpha_2 \alpha_3}+\frac{x_1 x_4}{\alpha_1 \alpha_4}\right) \\ &\quad =\overline{12}\,\overline{34}\left(\frac{x_4}{\alpha_4}-\frac{x_2}{\alpha_2}\right)\left(\frac{x_3}{\alpha_3}-\frac{x_1}{\alpha_1}\right)+\overline{13}\,\overline{42}\left(\frac{x_2}{\alpha_2}-\frac{x_1}{\alpha_1}\right)\left(\frac{x_4}{\alpha_4}-\frac{x_3}{\alpha_3}\right) \\ &\quad =\overline{12}\,\overline{13}\,\overline{34}\,\overline{42}\,[t_{12}\,t_{34}-t_{13}\,t_{42}] \equiv 0. \end{aligned}$$

But the vanishing of the matrix is also the condition that constants $\varkappa$, λ, μ, ν can be found such that

$$\varkappa \Sigma \frac{x_i}{\alpha_i} z_i + \lambda \Sigma \frac{y_i}{\alpha_i} z_i + \mu \Sigma \frac{x_i y_i}{\alpha_i^2} z_i + \nu \Sigma z_i \equiv 0;$$

i. e., that the polar space of $\varkappa x + \lambda y$ as to Q is the polar space of xy as to $C_3^{(n-2)}$.

The n-space Π_i determines an n-point, one of the points being $-(n-1), 1, 1, \ldots, 1$. This n-point is perspective to the n-point p_i. The center of perspection is the point $\alpha_1 - \frac{1}{n}\alpha$, $\alpha_2 - \frac{1}{n}\alpha$, $\ldots$, $\alpha_n - \frac{1}{n}\alpha$. Dually the n-point p_i determines an n-space perspective to the n-space Π_i and the S_{n-3}

of perspection is $\sum_1^n \frac{x_i}{\alpha_i} = 0$. The complete Desargues configuration determines $n+1$ such S_{n-3}'s. The $C_3^{(n-2)}$ is defined by the requirements that it have the orthic n-space Π_i whose poles as to $C_3^{(n-2)}$ are points p_i.

(122) *Given a quadric* (Q) *apolar to all the point quadrics on a norm-curve* (N_{n-2}) *in* S_{n-2}, *then there exists* ∞^3 *n-points on the norm-curve which are orthic to the quadric and which lie in an* $I_{3,n-3}$. *If one of these n-points* (p_i) *be isolated, its polar orthic n-space* (Π_i) *as to the quadric determines with it a unique* S_{n-3} $\left(\Sigma \frac{x_i}{\alpha_i} = 0\right)$. *This* S_{n-3} *is the polar space of the quadric as to a unique cubic spread* $(C_3^{(n-2)})$ *which has the same orthic n-space with the same poles. The norm-curve is an apolarity involution curve of this spread and the* $I_{2,n-2}$ *of anthorthic sets on it is contained in the* $I_{3,n-3}$.

§ 15. *The Perspective Curves of Stahl.*

In an article* entitled "Zur Erzeugung der ebenen rationalen Curven," W. Stahl develops some properties of rational envelopes, $E_m^{(2)}$, perspective to a given rational curve, $R_n^{(2)}$. His mode of treatment applies only to envelopes whose class, m, is less than n. The method here given† is quite general, though it appears that a natural upper limit for m is $2(n-1)$, the class of $R_n^{(2)}$.

Let $u_i = (\lambda_i \tau)^m$ and $x_i = (l_i t)^n$, $i = 1, 2, 3$, be respectively the rational envelope $E_m^{(2)}$ and the rational curve $R_n^{(2)}$ in parametric form. The (m, n) relation between τ and t,

$$(\lambda_1 \tau)^m (l_1 t)^n + (\lambda_2 \tau)^m (l_2 t)^n + (\lambda_3 \tau)^m (l_3 t)^n = 0,$$

expresses that the line τ of E_m and the point t of R_n are incident. The relation expanded in powers of (τt) is

$$(123)\quad \left\{\sum_1^3 (\lambda_i s)^m (l_i s)^n\right\}_{\tau^m t^n} + \frac{m n}{m+n}(\tau t)\left\{\sum_1^3 (\lambda_i l_i)(\lambda_i s)^{m-1}(l_i s)^{n-1}\right\}_{\tau^{m-1} t^{n-1}}$$
$$+ \frac{\binom{m}{2}\binom{n}{2}}{\binom{m+n-1}{2}}(\tau t)^2 \left\{\sum_1^3 (\lambda_i l_i)^2 (\lambda_i s)^{m-2}(l_i s)^{n-2}\right\}_{\tau^{m-2} t^{n-2}} + \ldots = 0.$$

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* *Mathematische Annalen*, Vol. XXXVIII (1891). See also an article by R. Schumacher in the same volume.

† Cf. F. Meyer, *Mathematische Annalen*, Vol. XXIX (1887), p. 447; and H. Brill: "Ueber rationale Curven und Regelflächen," *Münchener Berichte*, 1886.

By definition *E_m and R_n are perspective if like-named line and point are incident.* The necessary and sufficient condition for this is that (τt) be a factor of (123) or that the $(m+n)$-ic, $\sum_1^3 (\lambda_i s)^m (l_i s)^n$, vanish identically. If $R_n^{(2)}$ be given, this imposes $m+n+1$ linear homogeneous conditions on the $3(m+1)$ constants in the forms $(\lambda_i \tau)^m$. Thus there is a system of ∞^{2m-n+1} rational envelopes of class m perspective to a given general $R_n^{(2)}$. The arbitrary constants of the system occur linearly in the parametric equation of the envelope. The multiplicity of the system increases with m, but there is in the general case a lower limit, $m \geqq \frac{n-1}{2}$.

Let $\sum_1^3 (\lambda_i s)^m (l_i s)^n \equiv 0$, and let s be a root of $\sum_1^3 (\lambda_i l_i)(\lambda_i s)^{m-1} (l_i s)^{n-1}$. The $(m-1)$-th polar of $\tau = s$ as to this $(m+n-2)$-ic has a root $t = s$, whence the line $\tau = s$ of E_m touches R_n at the point $t = s$. Since the perspective relation is dual, the point $t = s$ of R_n lies on E_m at the contact of $\tau = s$; *i. e.*, E_m and R_n touch with the common element, $\tau = t = s$. The requirement that the $(m+n-2)$-ic have a root s is a linear condition on its coefficients. Thus, as long as $2m-n+1 \leqq m+n-2$, or $m < 2(n-1)$, E_m can be uniquely determined by assigning $2m-n+1$ of its $m+n-2$ contacts with R_n. The remaining $2n-m-3$ contacts are then determined. Hence the sets of contacts of E_m's with R_n lie in an involution.

(124) *The rational curve R_n has ∞^{2m-n+1} perspective rational envelopes E_m which, when expressed parametrically, lie in a linear system. Each E_m touches R_n at $m+n-2$ points. If $m < 2(n-1)$, these contacts lie in an involution, $I_{2m-n+1,\, 2n-3-m}$.*

Since the expansion (123) is linear, the parametric sum of two perspective E_m's is a perspective E_m.

Let us ask further under what conditions the contacts of E_m can be assigned so that its class reduces to $m-1$ due to the presence of a common linear factor in the m-ics, $(\lambda_i \tau)^m$. Let $(\lambda_i \tau)^m = (\rho \tau) \cdot (\mu_i \tau)^m$. Since the E_{m-1}, $u_i \equiv (\mu_i \tau)^{m-1}$, is also to be perspective to R_n, $\sum_1^3 (\mu_i s)^{m-1} (l_i s)^n \equiv 0$. Then E_{m-1} is completely determined by giving $2m-n-1$ of the contacts $\sum_1^3 (\mu_i l_i)(\mu_i s)^{m-2} (l_i s)^{n-1} = 0$.

On the other hand, the incidence condition of the degenerate E_m is

$$(\rho\tau)\sum_1^3(\mu_i\tau)^{m-1}(l_it)^n=0.$$

Its expansion is

$$\left\{(\rho s).\sum_1^3(\mu_is)^{m-1}(l_is)^n\right\}_{\tau^m t^n}+\frac{mn}{m+n}(\tau t)\left\{\frac{m-1}{n}(\rho s)\sum_1^3(\mu_il_i)(\mu_is)^{m-2}(l_is)^{n-1}\right.$$
$$\left.+\frac{1}{m}\sum_1^3(\rho l_i)(\mu_is)^{m-1}(l_is)^{n-1}\right\}_{\tau^{m-1}t^{n-1}}+\cdots\cdot$$

Since $\sum\limits_1^3(\mu_is)^{m-1}(l_is)^n\equiv 0$, the degenerate E_m must occur in the family of perspective E_m's. By polarizing as to ρ the identity just written and from the usual symbolic identity, two new identities are obtained,

$$\sum_1^3(m-1)(\mu_i\rho)(\mu_is)^{m-2}(l_is)^n+\sum_1^3 n(l_i\rho)(\mu_is)^{m-1}(l_is)^{n-1}\equiv 0,$$

$$\sum_1^3(\mu_i\rho)(\mu_is)^{m-2}(l_is)^n-\sum_1^3(l_i\rho)(\mu_is)^{m-1}(l_is)^{n-1}\equiv(\rho s).\sum_1^3(\mu_il_i)(\mu_is)^{n-2}(l_is)^{n-1},$$

by means of which the second term of the expansion of the degenerate E_m can be written as

$$\frac{n(m-1)}{n+m-1}(\tau t)\left\{(\rho s).\sum_1^3(\mu_il_i)(\mu_is)^{m-2}(l_is)^{n-1}\right\}_{\tau^{m-1}t^{n-1}}.$$

If, therefore, of the $2m-n+1$ arbitrary contacts of E_m, $2m-n$ are chosen which are the contacts of a perspective E_{m-1}, then, however the remaining contact (ρ) is chosen, the E_m so determined is actually the perspective E_{m-1}. Thus the $I_{2m-n+1,\,2n-m-3}$ of contacts of perspective E_m's contains as neutral sets the $m+n-3$ contacts of perspective E_{m-1}'s.

The E_{2n-3}'s have $3n-5$ contacts which can be chosen at random; the $E_{2(n-2)}$'s have $3(n-2)$ contacts which lie in an $I_{3n-7,\,1}$, whence these contacts form sets of $3(n-2)$ points on R_n whose parameters are apolar to a form, P, of order $3(n-2)$. From the well-known properties of neutral sets we have

(125) *The* $3n-5$ *contacts of an* E_{2n-3} *perspective to the general* R_n *determine the* E_{2n-3}. *There is a set,* P, *of* $3(n-2)$ *points on* R_n *at each of which a single perspective* $E_{2(n-2)}$ *has a contact of order* $3(n-2)$ *with* R_n. *Every perspective* $E_{2(n-2)-k}$ *touches* R_n *at* $3(n-2)-k$ *points whose parameters are an apolar set of the form* P.

To obtain the parametric equation of the perspective E_m, let E_m and R_n be respectively

$$\begin{aligned} u_1 &= \alpha_0 \tau_1^m + \alpha_1 \tau_1^{m-1}\tau_2 + \dots + \alpha_m \tau_2^m, \\ u_2 &= \beta_0 \tau_1^m + \beta_1 \tau_1^{m-1}\tau_2 + \dots + \beta_m \tau_2^m, \\ u_3 &= \gamma_0 \tau_1^m + \gamma_1 \tau_1^{m-1}\tau_2 + \dots + \gamma_m \tau_2^m, \end{aligned}$$

and

$$\begin{aligned} x_1 &= (a t)^n = a_0 t_1^n + a_1 t_1^{n-1} t_2 + a_2 t_1^{n-2} t_2^2 + \dots + a_n t_2^n, \\ x_2 &= (b t)^n = b_0 t_1^n + b_1 t_1^{n-1} t_2 + b_2 t_1^{n-2} t_2^2 + \dots + b_n t_2^n, \\ x_3 &= (c t)^n = c_0 t_1^n + c_1 t_1^{n-1} t_2 + c_2 t_1^{n-2} t_2^2 + \dots + c_n t_2^n. \end{aligned}$$

Let also $(a t)^{n-1} = a_0 t_1^{n-1} + a_1 t_1^{n-2} t_2 + \dots + a_{n-1} t_2^{n-1}$, etc.; let $s^{(1)}, s^{(2)}, \dots, s^{(2m-n+1)}$ be the parameters of the assigned contacts; and let $(ux)=0$ be the line of E_m with parameter τ. The perspective condition and the given contacts fix the ratios α_i, β_1, γ_i. The additional equation, $(ux)=0$, permits of their elimination by means of a determinant of order $3(m+1)$. A slight modification of this determinant gives these results:

The perspective point of an R_2 with contact $s^{(1)}$ is

$$\begin{vmatrix} a_0 & b_0 & c_0 & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & a_0 & b_0 & c_0 \\ a_2 & b_2 & c_2 & a_1 & b_1 & c_1 \\ 0 & 0 & 0 & a_2 & b_2 & c_2 \\ 0 & 0 & 0 & 2s_2^{(1)}(a s^{(1)}) & 2s_2^{(1)}(b s^{(1)}) & 2s_2^{(1)}(c s^{(1)}) \\ \tau_1 x_1 & \tau_1 x_2 & \tau_1 x_3 & \tau_2 x_1 & \tau_2 x_2 & \tau_2 x_3 \end{vmatrix} = 0.$$

The unique perspective point of an R_3 is

$$\begin{vmatrix} a_0 & b_0 & c_0 & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & a_0 & b_0 & c_0 \\ a_2 & b_2 & c_2 & a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 & a_2 & b_2 & c_2 \\ 0 & 0 & 0 & a_3 & b_3 & c_3 \\ \tau_1 x_1 & \tau_1 x_2 & \tau_1 x_3 & \tau_2 x_1 & \tau_2 x_2 & \tau_2 x_3 \end{vmatrix} = 0.$$

The perspective conic of an R_3 with contacts $s^{(1)}$ and $s^{(2)}$ is

$$\begin{vmatrix} a_0 & b_0 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & a_0 & b_0 & c_0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & a_1 & b_1 & c_1 & a_0 & b_0 & c_0 \\ a_3 & b_3 & c_3 & a_2 & b_2 & c_2 & a_1 & b_1 & c_1 \\ 0 & 0 & 0 & a_3 & b_3 & c_3 & a_2 & b_2 & c_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_3 & b_3 & c_3 \\ 0 & 0 & 0 & (as^{(1)})^3 & (bs^{(1)})^3 & (cs^{(1)})^3 & 2s_2^{(1)}(as^{(1)})^2 & 2s_2^{(1)}(bs^{(1)})^3 & 2s_2^{(1)}(cs^{(1)})^2 \\ 0 & 0 & 0 & (as^{(2)})^3 & (bs^{(2)})^3 & (cs^{(2)})^3 & 2s_2^{(2)}(as^{(2)})^2 & 2s_2^{(2)}(bs^{(2)})^2 & 2s_2^{(2)}(cs^{(2)})^2 \\ \tau_1^2 x_1 & \tau_1^2 x_2 & \tau_1^2 x_3 & \tau_1\tau_2 x_1 & \tau_1\tau_2 x_2 & \tau_1\tau_2 x_3 & \tau_2^2 x_1 & \tau_2^2 x_2 & \tau_2^2 x_3 \end{vmatrix} = 0$$

Let $\{\delta\}$ denote the matrix $\begin{Vmatrix} a_0\, b_0\, c_0 \\ \cdots\cdots \\ a_n\, b_n\, c_n \end{Vmatrix}$ of the coefficients; let $\|\underline{\{\delta\}\ \{\delta\}\ \{\delta\}\cdots}\|$ denote the matrix formed by placing a series of matrices $\{\delta\}$ side by side, each one line lower than the preceding one, the elements not otherwise provided for being zero; let a matrix like $\|\, 2s_2(as)^2\ 2s_2(bs)^2\ 2s_2(cs)^2 \,\|$ be denoted by $2s_2|(as)^2|$, and a matrix like $\|\tau_1^2 x_1\ \tau_1^2 x_2\ \tau_1^2 x_3\|$ be denoted by $\tau_1^2|x|$. In this new notation the three determinants just written are

$$\left|\begin{array}{cc} \{\delta\} & \{\delta\} \\ \hline |0| & 2s_2|(as)| \\ \tau_1|x| & \tau_2|x| \end{array}\right| = 0, \quad \left|\begin{array}{cc} \{\delta\} & \{\delta\} \\ \hline \tau_1|x| & \tau_2|x| \end{array}\right| = 0, \quad \left|\begin{array}{ccc} \{\delta\} & \{\delta\} & \{\delta\} \\ \hline |0| & |(as^{(1)})^3| & 2s_2^{(1)}|(as^{(1)})^2| \\ |0| & |(as^{(2)})^3| & 2s_2^{(2)}|(as^{(2)})^2| \\ \tau_1^2|x| & \tau_1\tau_2|x| & \tau_2^2|x| \end{array}\right| = 0.$$

The $I_{2,1}$ of contacts of perspective conics of an R_3 is

$$\left|\begin{array}{ccc} \{\delta\} & \{\delta\} & \{\delta\} \\ \hline |0| & |(as^{(1)})^3| & 2s_2^{(1)}|(as^{(1)})^2| \\ |0| & |(as^{(2)})^3| & 2s_2^{(2)}|(as^{(2)})^2| \\ |0| & |(as^{(3)})^3| & 2s_2^{(3)}|(as^{(3)})^2| \end{array}\right| = 0.$$

Similarly the equations of the perspective conics, cubics, quartics and quintics of an R_4 are

$$\left|\begin{array}{ccc} \{\delta\} & \{\delta\} & \{\delta\} \\ \hline |0| & |(as^{(1)})^4| & 2s_2^{(1)}|(as^{(1)})^3| \\ \tau_1^2|x| & \tau_1\tau_2|x| & \tau_2^2|x| \end{array}\right| = 0,$$

$$\left|\begin{array}{cccc} \{\delta\} & \{\delta\} & \{\delta\} & \{\delta\} \\ \hline |0| & s_1^{(1)}|(as^{(1)})^4| & 2s_2^{(1)}|(as^{(1)})^4| & 3s_2^{(1)2}|(as^{(1)})^3| \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \tau_1^3|x| & \tau_1^2\tau_2|x| & \tau_1\tau_2^2|x| & \tau_2^3|x| \end{array}\right| = 0,$$

$$\left|\begin{array}{ccccc} \{\delta\} & \{\delta\} & \{\delta\} & \{\delta\} & \{\delta\} \\ \hline |0| & s_1^{(1)2}|(as^{(1)})^4| & 2s_1^{(1)}s_2^{(1)}|(as^{(1)})^4| & 3s_2^{(1)2}|(as^{(1)})^4| & 4s_2^{(1)3}|(as^{(1)})^3| \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \tau_1^4|x| & \tau_1^3\tau_2|x| & \tau_1^2\tau_2^2|x| & \tau_1\tau_2^3|x| & \tau_2^4|x| \end{array}\right| = 0,$$

$$\left|\begin{array}{cccccc} \{\delta\} & \{\delta\} & \{\delta\} & \{\delta\} & \{\delta\} & \{\delta\} \\ \hline |0| & s_1^{(1)3}|(as^{(1)})^4| & 2s_1^{(1)2}s_2^{(1)}|(as^{(1)})^4| & 3s_1^{(1)}s_2^{(1)2}|(as^{(1)})^4| & 4s_2^{(1)3}|(as^{(1)})^4| & 5s_2^{(1)4}|(as^{(1)})^3| \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \tau_1^5|x| & \tau_1^4\tau_2|x| & \tau_1^3\tau_2^2|x| & \tau_1^2\tau_2^3|x| & \tau_1\tau_2^4|x| & \tau_2^5|x| \end{array}\right| = 0,$$

where in each case a sufficient number of contacts, $s^{(1)}, s^{(2)}, \ldots,$ must be given to complete the determinant. In each case also the involution of contacts is obtained by assigning an additional contact and dropping the condition $(ux)=0$; *i. e.*, the last row is replaced by one like those that immediately precede it.

The above examples suffice to indicate the equation of the general E_m perspective to R_n, $m<2(n-1)$. We have further that

(126) *The vanishing of the matrix* $\|\{\delta\}\,\{\delta\}\,\ldots\,\{\delta\}\|$ *formed from* ρ *matrices* $\{\delta\}$ *gives the single condition that an* $R_{2\rho}$, *or the conditions that an* $R_{2\rho+\sigma}$, *have a perspective* $E_{\rho-1}$.

The degree of the contact involutions in the combinant coefficients, δ_i, *i. e.*, in the three-row minors of $\{\delta\}$, is seen to be $m+1$, whence

(127) *The degree of the form* P *in* δ_i *is* $2n-3$. *Its apolar forms of order* $3(n-2)-j$ *lie in an involution of degree* $2n-3-j$ *in* δ_i.

Returning to the original incidence condition (123), let the coefficients of both $(\tau t)^0$ and $(\tau t)^1$ vanish. Then E_m and R_n will be called "doubly perspective." They must touch at every point of R_n and are merely dual forms of the same locus. If then, we require that R_n have a doubly perspective $E_{2(n-1)-1}$, this imposes the condition that R_n have a cusp. The equations of condition number $(m+n+1)+(m+n-1)=6(n-1)$. The $6(n-1)$ coefficients of $E_{2(n-1)-1}$ can be eliminated from them by means of order $6(n-1)$, which is the cusp condition of degree $2(n-1)$ in δ_i. For the R_3 this determinant is easily manipulated into the form

$$\left|\begin{array}{cccc}
\{\delta\} & \{\delta\} & \{\delta\} & \{\delta\} \\
\hline
|0| & |a_0| & |0| & |0| \\
|0| & |a_1| & 2\,|a_0| & |0| \\
|0| & |a_2| & 2\,|a_1| & 3\,|a_0| \\
|0| & |a_3| & 2\,|a_2| & 3\,|a_1| \\
|0| & |0| & 2\,|a_3| & 3\,|a_2|
\end{array}\right| = 0,$$

which again can be indicated more briefly by

$$\left|\begin{array}{cccc}
\{\delta\} & \{\delta\} & \{\delta\} & \{\delta\} \\
\hline
 & \{\delta\} & 2\{\delta\} & 3\{\delta_3\} \\
\hline
\end{array}\right| = 0,$$

where $\{\delta_3\}$ denotes the matrix $\{\delta\}$ with the row $|a_3|$ deleted. In the general case we find that

(128) *The cusp condition of R_n is the vanishing of the determinant*

$$\left|\begin{array}{cccccc} \{\delta\} & \{\delta\} & \{\delta\} & \dots\dots & \{\delta\} & \{\delta\} \\ \hline & \{\delta\} & 2\{\delta\} & \dots\dots & (2n-4)\{\delta\} & (2n-3)\{\delta_n\} \\ \hline \end{array}\right|$$

of degree $2(n-1)$ in δ_i.

This is included in the more general theorem:

(129) *If R_n be actually of order n (i. e., if the n-ics $(l_i t)^n$ have no common factor), the condition that it have h cusps or their equivalent is the vanishing of the matrix*

$$\left\|\begin{array}{cccccc} \{\delta\} & \{\delta\} & \{\delta\} & \dots\dots & \{\delta\} & \{\delta\} \\ \hline & \{\delta\} & 2\{\delta\} & \dots\dots & (2n-4-h)\{\delta\} & (2n-3-h)\{\delta\} \\ \hline \end{array}\right\|,$$

which expresses that it has a doubly perspective $E_{2(n-1)-h}$.

The existence of doubly perspective envelopes of class $m \geqq 2(n-1)$ explains why the contacts alone are no longer sufficient to determine the envelope. Thus if R_n in lines is $u_i = (\mu_i \tau)^{2(n-1)}$ and E_m is a perspective envelope of class $m = 2(n-1) + k$, then the ∞^{k+1} envelopes $u_i = (\lambda_i \tau)^m + (\rho \tau)^k (\mu_i \tau)^{2(n-1)}$ obtained by varying the coefficients of $(\rho \tau)^k$ are all perspective and have the same contacts as E_m.

If, from the incidence condition (123), the factor (τt) be removed, the resulting $(m-1, n-1)$ relation between τ and t has a certain number of involutive pairs which are the pairs of parameters of the double points of R_n and the double tangents of E_m. They are found by eliminating τ from the two equations $(\lambda \tau)^{m-1} (l t)^{n-1} = 0$ and $(\lambda t)^{m-1} (l \tau)^{n-1}$. The resultant is of order $(m-1)^2 + (n-1)^2$ in t and gives the $(m-1)(m-2) + (n-1)(n-2)$ double point and double tangent parameters and the $m+n-2$ contact parameters which are degenerate involutive pairs. Since the double point parameters always satisfy the $(m-1, n-1)$ relation, if R_n has a cusp, the cusp parameter satisfies the relation and counts as a contact; whence

(130) *If E_m and R_n are perspective, E_m passes through all the cusps of R_n and R_n touches all the flex-tangents of E_m.*

The form P of order $3(n-2)$ and of degree $2n-3$ in δ_i has apolar forms of order $3(n-2)-j$ which lie in an involution of degree $j+1$ in the coefficient of P and therefore of degree $(2n-3)(j+1)$ in δ_i. According to (127) the actual degree of this involution is $2n-3-j$. The outstanding factor of degree $2(n-1)j$ must be the j-th power of the cusp condition. For, if R_n has a cusp, the cusp is a contact of all perspective E_m's. It must therefore be the contact of those envelopes of class $2(n-2)$ whose contacts all coincide at the points, $P=0$ (125). Hence $P=0$ is the cusp parameter repeated $3(n-2)$ times, and the contact involutions when formed directly from P must vanish identically. These involutions are now (disregarding the cusp contact) apolar forms of a form P_1 of order $3(n-2)-2$ and of degree $(2n-3)-1$ in δ_i. Hence

(131) *If R_n has a cusp, the covariant P is the cusp parameter repeated $3(n-2)$ times. If R_n has two cusps, P vanishes. If R_n has h cusps, the contact involutions of perspective envelopes are (omitting the improper cusp contacts) the involutions of forms apolar to a covariant P_h of order $3(n-2)-2h$ and of degree $(2n-3)-h$ in δ_i.*

The following fairly evident theorem furnishes the connection with the preceding § 14.

(132) *The $(m-1, n-1)$ relation, $(\lambda\tau)^{m-1}(lt)^{n-1}=0$, can be obtained from perspective E_m and R_n if $m-2$ linearly independent forms $(\mu^k\tau)^m$ exist such that $(\mu_k t)(\mu_k\lambda)^{m-1}(lt)^{n-1}\equiv 0$, or also if $n-2$ linearly independent forms $(m_k t)^n$ exist such that $(\lambda\tau)^{m-1}(lm_k)^{n-1}(m_k\tau)\equiv 0$. From a relation satisfying the given conditions E_m and R_n can be thus constructed: With regard to a norm-curve N_m in S_m, the $m-2$ forms, $(\mu_k\tau)^m$, are represented by $m-2$ S_{m-1}'s which meet in a plane E. The S_{m-1}'s, τ, of N_m cut E in the lines, τ, of an E_m. If τ and t satisfy the given relation, the two S_{m-1}'s of N_m, τ and t, meet on E in the point t of a rational R_n perspective to E_m. In particular any $(1, n-1)$ relation determines an R_n referred to its perspective conic.*

(133) *The two classes of $R_n^{(2)}$'s for whose apolarity cubic spreads the involution curves were determined in § 14 are, respectively, the $R_n^{(2)}$ with a perspective point (an $(n-1)$-fold point), and the $R_n^{(2)}$ with a perspective conic.*

The foregoing has comparatively little in common with the work of Stahl. Before leaving the subject of perspective curves it may be worth while to develop some of his results in a much simpler way.

The equation of a line through the points t_1 and t_2 of the given R_n is

$$(134)\qquad \Gamma = \begin{vmatrix} (l_1 t_1)^n & (l_2 t_1)^n & (l_3 t_1)^n \\ (l_1 t_2)^n & (l_2 t_2)^n & (l_3 t_2)^n \\ x_1 & x_2 & x_3 \end{vmatrix} \frac{1}{(t_1 t_2)}$$

$$\equiv (\alpha x)(a_1 t_1)^{n-1}(a_2 t_2)^{n-1} = (\alpha x)(a_1 t_2)^{n-1}(a_2 t_1)^{n-1} = 0.$$

If, in $\Gamma = 0$, t_1 be the parameter of a fixed point p_1 of R_n and t_2 that of a variable point p_2, then, as t_2 varies, the line Γ describes $n-1$ times the fixed point p_1. Thus $\Gamma = 0$ is a parametric equation of the degenerate envelope p_1^{n-1}. Moreover, this envelope is perspective to R_n since the line $\tau = t_2$ contains the point $t = t_2$ of R_n. As p_1 runs over R_n, ∞^1 such degenerate envelopes are obtained. By forming their general linear combination, Stahl's general perspective E_{n-1} is obtained. The process of forming such a sum is merely that of forming the apolarity invariant of a general $(n-1)$-ic, $(b\,t)^{n-1}$, with Γ of order $n-1$ in t, whence, in terms of (134), the equation of the general perspective E_{n-1} is

$$(a\,x)(a_1 b)^{n-1}(a_2 \tau)^{n-1} = 0. \tag{135}$$

This is the equation of a line which, for given τ, passes through the point $t = \tau$ of R_n and cuts R_n again in $n-1$ points whose parameters are apolar to $(b\,t)^{n-1}$. Hence

(136) *The lines which cut R_n in $n-1$ points whose parameters are apolar to an arbitrarily chosen $(n-1)$-ic, $(b\,t)^{n-1}$, envelop a perspective E_{n-1}. If, however, $(b\,t)^{n-1}$ is the polar as to t_1 of a set F^n of the fundamental involution, E_{n-1} is made up of the lines through t_1 on R_n and a perspective envelope E_{n-2} which is independent of the choice of t_1; if $(b\,t)^{n-1}$ is the polar as to t_1 and t_2 of a form F^{n+1} whose first polars all are sets of the fundamental involution, E_{n-1} is made up of lines on t_1, lines on t_2, and a perspective E_{n-2} which is independent of t_1 and t_2, i. e., which depends only on F^{n+1}; etc.*

For it is quite clear that the lines on t_1 (on t_1 or t_2, etc.) cut R_n in $(n-1)$-ics apolar to $(b\,t)^{n-1}$; but there is also a further system of ∞^1 lines which cut R_n in $(n-1)$-ics which are canonizant sets of F^n (F^{n+1}, etc.); these $(n-1)$-ics are apolar to $(bt)^{n-1}$ whatever be t_1 (t_1 and t_2, etc.).

Returning to the degenerate E_{n-1}, a point of R_n repeated $n-1$ times, its contacts are the point itself and the $2(n-2)$ contacts of tangents from the point. If the point is a double point, t_1, t_1' on R_n, then another factor $(t_1' t_2)$ separates from Γ. The E_{n-1} becomes an E_{n-2} whose contacts are at the double point and at the $2(n-3)$ contacts of tangents from the double point. Since these contacts are apolar to P, we have that

(137) *The parameters of a node and the $2(n-3)$ parameters of tangents from the node are apolar to the covariant P of order $3(n-2)$. Thus only $n-2$ of the $\frac{1}{2}(n-1)(n-2)$ such $2(n-2)$-ics are linearly independent.*

§ 16. *The Gordan Expansion of the Form $H_{2,3}$.*

The expansion of $H_{2,3}=(a_1x_1)^3(a_2x_2)^3(a_3x_3)^3$ is a sum of terms each of which is a product of identical covariants, (x_ix_k), and of a polarized form. The polarized form must be a linear covariant of $H_{2,3}$. Because of the equivalence of the symbols, a_1, a_2, a_3 in $H_{2,3}$, its linear covariants must be the same symbolically as the covariants of the third degree of a binary cubic,

$$(a_1x)^3=(a_2x)^3=(a_3x)^3.$$

These are the cube of the cubic, the product of the cubic and its Hessian, and the cubicovariant. Hence the required linear covariants of $H_{2,3}$ are:

$$\begin{aligned}(\pi x)^9&=(a_1x)^3(a_2x)^3(a_3x)^3,\\ (\rho x)^5&=(a_1a_2)^2(a_1x)(a_2x)(a_3x)^3,\\ (\sigma x)^3&=(a_1a_2)^2(a_3a_1)(a_2x)(a_3x)^2.\end{aligned}$$

Some products which occur in the expansion are:

$$\begin{aligned}p_1&=(x_2x_3)^2(\rho x_2)(\rho x_3)(\rho x_1)^3,\\ p_2&=(x_3x_1)^2(\rho x_3)(\rho x_1)(\rho x_2)^3,\\ p_3&=(x_1x_2)^2(\rho x_1)(\rho x_2)(\rho x_3)^3,\end{aligned}$$

$$\begin{aligned}r_1&=(x_2x_3)^2(x_2x_1)\,.\,(\sigma x_3)(\sigma x_1)^2, & s_1&=(x_2x_3)^2(x_3x_1)\,.\,(\sigma x_2)(\sigma x_1)^2,\\ r_2&=(x_3x_1)^2(x_3x_2)\,.\,(\sigma x_1)(\sigma x_2)^2, & s_2&=(x_3x_1)^2(x_1x_2)\,.\,(\sigma x_3)(\sigma x_2)^2,\\ r_3&=(x_1x_2)^2(x_1x_3)\,.\,(\sigma x_2)(\sigma x_2)^2, & s_3&=(x_1x_2)^2(x_2x_3)\,.\,(\sigma x_1)(\sigma x_3)^2.\end{aligned}$$

In terms of these the expansion is:

$$\begin{aligned}(138)\quad (a_1x_1)^3(a_2x_2)^3(a_3x_3)^3=(\pi x_1)^3(\pi x_2)^3(\pi x_3)^3&+\tfrac{9}{16}(p_1+p_2+p_3)\\ &-\tfrac{9}{35}(r_1+r_2+r_3+s_1+s_2+s_3).\end{aligned}$$

If $H_{2,3}$ be interpreted as the apolarity equation of a cubic surface, C_3, with regard to an N_3, then $(\pi x)^9=0$ is the equation of the parameters on N_3 of the meets of N_3 and C_3. If $(\pi x)^9\equiv 0$, then N_3 lies on C_3. In this case a tangent plane of C_3 at a point t on N_3 touches N_3 at t and meets it again at τ. The relation between t and τ is $(a_1\tau)^3(a_2t)^3(a_3t)^3=0$, which now becomes

$$(139)\qquad 9(\tau t)^2\{\tfrac{1}{3}(\rho\tau)(\rho t)^4-\tfrac{2}{35}(\tau t)(\sigma t)^3\}=0.$$

τ and t coincide at a point where N_3 touches an asymptotic curve of C_3, say at an "asymptotic point" of N_3. Thus if $(\pi x)^9\equiv 0$, $(\rho x)^5=0$ determines the five asymptotic points on N_3. If also $(\rho x)^5 \equiv 0$, N_3 is an asymptotic curve of C_3 which has double points at the three points, $(\sigma x)^3 = 0$, of N_3.

Denote the cubic surface by D_3 if it contains N_3. Let the apolarity form of D_3 be $(d_1 x_1)^3 (d_2 x_2)^3 (d_3 x_3)^3$, whence $(d_1 \tau)^3 (d_2 t)^3 (d_3 t)^3 = (\tau t)^2 \,.\, (\lambda \tau)(l t)^4$. By comparison with (139) one sees that the (1, 4) form $(\lambda\tau)(lt)^4$ can replace the apolarity form since it determines $(\rho x)^5$ and $(\sigma x)^3$. It is also evident that

(140) *The condition that D_3 have one, two, three, or four double points on N_3 is that from $(\lambda\tau)(lt)^4$ there shall separate respectively one, two, three, or four factors linear in t. If both $(\lambda\tau)(lt)^4 = 0$ and $(\lambda t)(l\tau)^4 = 0$, the join of t and τ is one of the six chords of N_3 on D_3. If $(\lambda\tau)(lt)^4$ contains a factor cubic in t, and if the other factor is symmetric in t and τ, then D_3 is the product of a quadric on N_3 and a plane which cuts N_3 in the points given by the cubic factor.*

Let us ask further under what conditions a quadric cone on N_3 with vertex t_1 shall touch D_3 along N_3. Necessarily then, if $\lambda_1 (t_1 x)^3 + \lambda_2 (t_2 x)^3$ [which, as t_2 runs along N_3, is for variable $\lambda_1 : \lambda_2$ a generator of the cone] be substituted in $(d_1 x_1)^3 (d_2 x_2)^3 (d_3 x_3)^3$, the term in $\lambda_1 \lambda_2^2$ vanishes, *i. e.*,

$$(d_1 t_1)^3 (d_2 t_2)^3 (d_3 t_2)^3 = (t_1 t_2)^2 \,.\, (\lambda t_1)(l t_2)^4 = 0.$$

Since $(\lambda t_1)(l t_2)^4$ vanishes for all values of t_2, the form $(\lambda\tau)(lt)^4$ must be $(\tau t_1) \,.\, (q t_2)^4$, whence

(141) *In order that a quadric cone on N_3 shall touch D_3 along N_3 it is necessary that D_3 shall have four double points on N_3. Conversely if D_3 has four double points and if N_3 lies on D_3 and its four double points, then the tangent planes of D_3 at points on N_3 meet N_3 again in a unique point, the vertex of a quadric cone on N_3 which touches D_3 along N_3.**

§17. *The Form $I_{2,3}$ and the Curve $R_5^{(2)}$.*

Let the $R_5^{(2)}$, $x_i = (l_i t)^5$, determine the $I_{2,3}$, $(a_1 x_1)^3 (a_2 x_2)^3 (a_3 x_3)^3 = 0$. The three quintics, $(m_i t)^5$, apolar to $(l_i t)^5$ determine the "apolar" or "fundamental" involution, $\overline{I}_{2,3}$. Thus an $R_5^{(2)}$ projectively determines an $\overline{R}_5^{(2)}$. Each involution is easily expressed in terms of the sets of the other; thus

$$\begin{aligned} I_{2,3} &= (m_1 m_2)(m_1 m_3)(m_2 m_3)\,\Pi\,(m_i x_k), \\ \overline{I}_{2,3} &= (l_1 l_2)(l_1 l_3)(l_2 l_3)\,\Pi\,(l_i x_k), \qquad i, k = 1, 2, 3. \end{aligned}$$

---

*If D_5 be mapped on a plane by means of cubic curves on the six points of a 4-line, (141) can be stated thus: Cubic curves on the six points of a 4-line with double points on a line l all meet l again in a single point p. It can be shown from other considerations that p is the point of contact of a conic which touches the 4-line and l.

Every covariant of one involution is a covariant of the other; in particular the flex-equations of $R_5^{(2)}$ and $\bar{R}_5^{(2)}$,

$$(m_1 m_2)(m_1 m_3)(m_2 m_3)\Pi(m_i x)^3 = 0$$

and

$$(l_1 l_2)(l_1 l_3)(l_2 l_3)\Pi(l_i x)^3 = 0,$$

are the same. The linear covariants,

$$(l_2 l_3)^5(l_1 x)^5 + (l_3 l_1)^5(l_2 x)^5 + (l_1 l_2)^5(l_3 x)^5$$

and

$$(m_2 m_3)^5(m_1 x)^5 + (m_3 m_1)^5(m_2 x)^5 + (m_1 m_2)^5(m_3 x)^5,$$

are the same; they are the set common to both involutions.

If $I_{2,3}$ be expanded in a Gordan series (138), the various terms contain polars of $(\pi x)^9$, $(\rho x)^5$, and $(\sigma x)^3$. Since these forms are the only linear covariants of $I_{2,3}$, $(\pi x)^9 = 0$ must be the flex-equation of $R_5^{(2)}$ and $(\rho x)^5 = 0$ must be the set common to $I_{2,3}$ and $\bar{I}_{2,3}$. If $\bar{I}_{2,3}$ be similarly expanded, its terms also must contain polars of the same forms. If the factors of proportionality be chosen so that $(\pi x)^9$ is the same in both expansions, the other forms can differ only in sign. But the identical vanishing of $(\rho x)^5$ is the condition that $I_{2,3}$ and $\bar{I}_{2,3}$ coincide. Hence

(142) *If in the Gordan expansion* (138) *of* $I_{2,3}$, *the sign of* $(\rho x)^5$ *be changed, the resulting expansion is that of the involution* $\bar{I}_{2,3}$.

Let h be a quadric on N_3 whose equation is $(x_1 x_2)^2(hx_1)(hx_2) = 0$; and let the plane equation of a point quadric, $(a_2 x_2)^3(a_3 x_3)^3 = 0$, be $(\alpha_2 x_2)^3(\alpha_3 x_3)^3 = 0$, (102). Then the associated cubic surface, C_h, (110), of the quadric h has the apolarity equation

$$C_h = (a_1 x_1)^3(a_1' x_2)^3(a_1'' x_3)^3(\alpha_2 \alpha_3)^2(\alpha_2 h)(\alpha_3 h) = 0.$$

We have seen (p. 342) that C_h must contain N_3, whence

$$(a_1 x)^3(a_1' x)^3(a_1'' x)^3(\alpha_2 \alpha_3)^2(a_2 h)(\alpha_3 h) = 0.$$

But if a cubic surface contains N_3 its apolarity equation can be replaced by a (1, 4) form (p. 360); *i. e.*,

$$[C_h]_{x_2=x_3} = (x_1 x_2)^2 \;.\; (p x_1)(q x_2)^4(r h)^2.$$

Again (p. 342), the intersection of h and C_h is N_3 and a residual cubic curve which meets N_3 in the five points of one of the 16 pentahedra of (109). Every quadric on N_3 cuts out such a residual cubic except the cones on N_3. If h be a cone on N_3 with vertex τ, the residual cubic must coincide with N_3, and h must touch C_h along N_3. This requires (141) that the (1, 4) form $[C_h]_{x_2=x_3}$ degenerate into a linear form which locates the vertex of the cone and a quartic form which locates four double points of C_h; *i. e.*, the form $(px_1)(qx_2)^4(r\tau)^2$ is merely the product $(\tau x_1) \cdot (\lambda\tau)(lx_2)^4$. As the vertex τ runs over N_3, the four double points of the associated cubic surface also run over N_3 and a unique (1, 4) form, $(\lambda\tau)(lt)^4$, associated with $I_{2,3}$, is determined, which is of the third degree in the coefficients of $I_{2,3}$.

In order to identify the form let r_1, r_2 be the parameters on N_3 of a pair of corresponding points of the Hessian of $C_3 = a_x^3$; let Q_{r_1} be the cone on N_3 with vertex r_1, and let P_{r_1} be the polar quadric of r_2 as to C_3 which is also a cone with vertex at r_1. Since there are ∞^1 triads, S_1, S_2, S_3, on N_3, which with r_1 and r_2 make up $A^5_{3,3}$'s of C_3, then $a_{r_2} a_{s_i} a_{s_k} = 0$. Since $a_{r_2} a_x^2$ is P_{r_1}, there are ∞^1 3-lines on Q_{r_1} self-conjugate as to P_{r_1}; *i. e.*, $(P_{r_1} P_{r_1} Q_{r_1} u)^2 \equiv 0$. Operating with this on a_x^3, we have

$$(P_{r_1} P_{r_1} Q_{r_1} a)^2 a_x = a_x b_{r_2} c_{r_2} (abc\, Q_{r_1})^2 \equiv 0.$$

This expresses that the associated cubic surface of the cone Q_{r_1} has a double point at r_2. Similarly the associated cubic surface of Q_{r_2} has a double point at r_1, whence r_1 and r_2 make up an involutive pair of the (1, 4) form just found. These results can be summarized thus:

(143) *The general $R_5^{(2)}$ has a unique perspective conic, K, whose incidence condition is a general* (1, 4) *form*, $(\lambda\tau)(lt)^4 = 0$. *In the representation of $R_5^{(2)}$ on N_3 this form has the following three interpretations:*

(a) *The apolarity involution curve N_3 of C_3 contains 5 poles as to C_3 of the Sylvester 5-plane of C_3. The 5-point and 5-plane are pole and polar as to a quadric Q. The polar quadric as to C_3 of a point τ has in common with Q a self-polar 4-point t on N_3.*

(b) *The associated cubic surface as to C_3 of a quadric cone on N_3 with vertex τ has four double points t on N_3.*

(c) *N_3 cuts the Hessian of C_3 in six pairs of corresponding points whose joins, L_i, belong to a double six of a cubic surface D_3 on N_3. The tangent plane of D_3 at a point t cuts N_3 again at a point τ.*

The (1, 4) *form is of the third degree in the coefficients of $I_{2,3}$. It is obtained from $(a_1 x_1)^3 (a_1' x_2)^3 (a_1'' x_3)^3 (\alpha_2 \alpha_3)^2 (\alpha_2 h)(\alpha_3 h)$ by putting $x_2 = x_3 = t$ and $(hx)^2 = (\tau x)^2$,* *and then removing the factors* $(x_1 t)^2 (x_1 \tau)$. *The* $I_{2,3}$ *is also of the third degree in the coefficients of the* (1, 4) *form. The extraneous invariant is the triple point condition.*

We have already remarked, that the (1, 4) form serves to replace the apolarity equation of a cubic surface on N_3. Thus the apolarity equation of D_3, a simultaneous covariant of C_3 and N_3, is known in terms of $(\lambda\tau)(lt)^4$.

(144) *The invariant theory of the general* $R_6^{(2)}$ *coincides with that of a general* (1, 4) *binary form.*

It is known that if either a given involution of n things or its apolar involution contains only one or only two independent forms, then the covariants of the involution can be expressed as covariants of a *single* general binary form. This includes the $I_{0,n}$ or $I_{n-1,1}$, in which case the single form is an n-ic; and the $I_{1,n-1}$ or $I_{n-2,2}$, in which case the single form is a $2(n-1)$-ic. The $I_{2,3}$ is the simplest involution not included in these cases and (144) gives the analogous theorem for the $I_{2,3}$. The advantage in discussing a form whose coefficients are subject to no relations is manifest.

If the cubic surface D_3 be mapped on a plane E so that the six lines L_i become points p_i, then N_3 maps into the $R_6^{(2)}$ with nodes at p_i. Quadrics on N_3 meet D_3 in residual cubic curves which map into the lines of E; in particular, the residual cubics which map into the lines of K are on the cones of N_3. Cubic surfaces on N_3 meet D_3 in residual sextics which map into quartic curves on the points p_i; in particular, plane sections of D_3 map into the adjoint cubics of $R_6^{(2)}$. The quartic developable of N_3 meets D_3 in a residual rational sextic which maps into K. Let x be a point of D_3, which is on the chord c_x through the points q_1 and q_2 of N_3. Let x map into a point y of E, and q_1 and q_2 into points q_1 and q_2 on $R_6^{(2)}$. Plane sections of D_3 on c_x map into the pencil of adjoint cubics of $R_6^{(2)}$ on q_1, q_2, and y. Since c_x is contained on the cones of N_3 with parameters q_1 and q_2, y is the meet of the tangents of K with parameters q_1 and q_2. Thus we have obtained Stahl's theorem: The pencil of adjoint cubics on two points of $R_6^{(2)}$ meet in a further point which is also the meet of the corresponding two tangents of K. Stahl's very pretty corollary—that a conic on five double points of $R_6^{(2)}$ contains ∞^1 3-points whose joins envelop K—can be applied to express the conditions on the six quadratics which determine the nodes of $R_6^{(2)}$. Only four of these can be chosen at random, leaving one degree of freedom for the other two. With reference to a norm-conic K, let the four given quadratics determine four points, p_1, p_2, p_3, p_4. Through these points there pass two conics C' and C'' which contain the vertices of triangles circumscribed to K. A tangent of K meets C' in two points whose further tangents to K meet at a point p_5 on C'. The same tangent meets C'' in two points whose further tangents to K meet at a point p_6 on C''. The points $p_1, \ldots, p_6$ determine on K the nodal parameters of an $R_6^{(2)}$; or there is an $R_6^{(2)}$ with double points $p_1, \ldots, p_6$ and with the perspective conic K.

The analogous construction for the six lines L_i of a double six is: Given four chords, L_1, L_2, L_3, L_4, of N_3, no three of which are on a quadric that contains N_3; if C' and C'' are the two transversals of $L_1, \ldots, L_4$, the planes through C' and C'', respectively, and any point of N_3 cut N_3 again in two further pairs of points whose joins are the lines L_5 and L_6.

BALTIMORE, *January* 25, 1910.

Systems of Tautochrones in a General Field of Force.[*]

By Harry Wilfred Reddick.

Introduction.

This paper deals with the system of curves each of which possesses the following property: A particle, starting from rest and moving along the curve, under the action of a given force, will reach a certain point, the center of tautochronous motion, in the same time from whatever point on the curve it starts. Friction is neglected and it is assumed that the acting force depends only on the position of the particle. The first part of the paper deals with plane systems. It is assumed that within the region considered the rectangular components of the force, $\phi(x, y)$, $\psi(x, y)$, are uniform functions which possess first and second derivatives and do not vanish identically.

In 1673 Huyghens found that when the acting force is gravity the tautochrones are cycloids with horizontal bases and concavity upwards. Since that time many extensions of the problem have been made and the tautochrones due to more complicated laws of force and resistance have been found. This paper, however, does not deal with the problem of finding the tautochrones due to a general positional force, but investigates the form of the differential equation and the derivable geometric properties of the system. This differential equation, which is of the third order and represents the total triply infinite system of tautochrones in the plane, is found in Article 1 from the well-known physical property of tautochrones, namely, that the tangential component of the acting force along the curve is proportional to the arcual distance of the moving particle from the center of tautochronous motion. The factor of proportionality must be negative for actual tautochrones, but, as it is eliminated in finding the differential equation of the system, the system includes virtual as well as actual tautochrones.

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For this reason we choose to include virtual tautochrones in the system under consideration.

The geometric properties of the system are next investigated. At any point (x, y) of the plane in a given direction y' there is a singly infinite system of tautochrones. If the osculating parabola of each of these curves is constructed at the given point, the locus of their foci will be a bicircular quartic having a flecnode at the given point, the stationary tangent at this point having the given direction y'. The angle θ between the two tangents to the quartic at the given point is related in a definite way to the angle θ' which the acting force makes with the stationary tangent, namely $\tan\theta = 3\tan\theta'$.

The inverse of the quartic with respect to the given point (x, y) is a hyperbola, one of whose asymptotes is the stationary tangent to the quartic at the point (x, y); the other asymptote is parallel to the second tangent to the quartic at the point (x, y). The locus of the center of this hyperbola as the given direction y' varies is a circular cubic having the given point (x, y) as double point; the real asymptote of the cubic is parallel to the direction of the force acting at the given point. A characteristic property of conservative forces is that the tangents to this cubic at the given point are perpendicular to each other.

The next property pertains to hyperosculating circles. If the osculating circle of each tautochrone through a given point (x, y) is constructed at this point, it is found that the locus of the centers of those circles which hyperosculate their respective tautochrones is a cubic through the given point. The tangent to this cubic at the given point is the line along which the force acts at the point.

In the next section the question of the sufficiency of the preceding geometric properties for characterizing a system of tautochrones is considered. These properties characterize the system of tautochrones through a given point, up to differential elements of the third order. Additional relations are found which make the characterization of the total triply infinite system of tautochrones complete.

In Articles 16–18, systems of curves of different types, depending on their possession of certain of the preceding geometric properties, are considered with regard to the effect of point transformations on them. It is found that the only point transformations which transform the system possessing all the geometric properties under consideration into a system of the same type are the similitude transformations. Articles 19–20 deal with the system of tautochrones corresponding to a given value of the time constant.

In the last part of the paper, space tautochrones are considered and the differential equation of the totality of tautochrones in three-dimensional space is found. The class of forces for which every straight line is a tautochrone is determined. It is found that a known relation between such a class of forces in the plane and the class of collineations does not hold in space.* Finally, it is shown that the only conservative forces for which every straight line is a tautochrone are those for which the potential is a quadratic function of x, y and z.

The Differential Equation of the Tautochrones.

1. Suppose that at each point of the plane a force is acting whose rectangular components are functions only of the coordinates of the point, namely $\phi(x, y)$, $\psi(x, y)$. Such a force is called a positional force. We shall consider the system of tautochrones in a plane, due to a positional force; that is, the system of curves each of which possesses the following property: A particle which, starting from rest, moves along the curve under the action of a given positional force, will reach a certain point, the center of tautochronous motion, in the same time from whatever point on the curve it starts.

For such a curve we have the well-known property,

$$T = k(s - s_0), \tag{1}$$

where T is the tangential component of the force, $s - s_0$ is the arc measured from the center of tautochronous motion to the position of the moving particle, and k is the time constant whose value in terms of τ, the time required for a particle to move from any point of the curve to the center, is

$$k = -\frac{\pi^2}{4\tau^2}.$$

By differentiating equation (1) twice with respect to s, we obtain the intrinsic equation of the system of all tautochrones in the plane,† namely

$$Nr_s = T_1 r^2 + (T_2 + \bar{N})r - T, \tag{2}$$

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* Kasner: "Tautochrones and Brachistochrones," *Bulletin of the American Mathematical Society*, Vol. XV (1909), p. 480.

† Kasner: "Tautochrones," *loc. cit.*, p. 477.

where r is the radius of curvature, T and N are respectively the tangential and normal components of the force, and

$$\left.\begin{aligned}
\bar{N} &= \frac{\psi_x + (\psi_y - \phi_x)\,y' - \phi_y\,y'^2}{1+y'^2}, \\
T_1 &= \frac{\phi_{xx} + (2\,\phi_{xy} + \psi_{xx})\,y' + (\phi_{yy} + 2\,\psi_{xy})\,y'^2 + \psi_{yy}\,y'^3}{(1+y'^2)^{\frac{3}{2}}}, \\
T_2 &= \frac{\phi_y + \psi_x + 2\,(\psi_y - \phi_x)\,y' - (\phi_y + \psi_x)\,y'^2}{1+y'^2},
\end{aligned}\right\} \tag{3}$$

subscripts denoting partial differentiation and primes denoting differentiation with respect to x. If we substitute in equation (2) the values of $\bar{N}$, T_1 and T_2 given in equations (3), and the values of N, T, r and r_s given by

$$\left.\begin{aligned}
r &= \frac{(1+y'^2)^{\frac{3}{2}}}{y''}, \quad & r_s &= \frac{3\,y'\,y''^2 - (1+y'^2)\,y'''}{y''^2}, \\
N &= \frac{\psi - y'\,\phi}{\sqrt{1+y'^2}}, \quad & T &= \frac{\phi + y'\,\psi}{\sqrt{1+y'^2}},
\end{aligned}\right\} \tag{4}$$

we obtain, after reduction, the Cartesian equation of the system of all tautochrones in the plane,

$$\left.\begin{aligned}
y''' = {} & \frac{[\phi_{xx} + (2\,\phi_{xy} + \psi_{xx})\,y' + (\phi_{yy} + 2\,\psi_{xy})\,y'^2 + \psi_{yy}\,y'^3]\,(1+y'^2)}{\phi\,y' - \psi} \\
& + \frac{\phi_y + 2\,\psi_x + 3\,(\psi_y - \phi_x)\,y' - (2\,\phi_y + \psi_x)\,y'^2}{\phi\,y' - \psi}\,y'' \\
& + \frac{3\,\phi\,y'^2 - 4\,\psi\,y' - \phi}{(\phi\,y' - \psi)(1+y'^2)}\,y''^2.
\end{aligned}\right\} \tag{5}$$

This is a differential equation of the third order and represents a triply infinite system of curves.

2. *Special case: Gravity.* When the acting force is gravity,

$$\phi = 0, \quad \psi = g.$$

Then equation (5) reduces to

$$y''' = \frac{4\,y'\,y''^2}{1+y'^2}. \tag{6}$$

Equation (6) represents the totality of tautochrones in the plane due to gravity. It is the differential equation of the system of cycloids with horizontal bases, and may be obtained by taking the Cartesian equation of such a system, namely,

$$x + c = a \cos^{-1}\left(1 \mp \frac{y+d}{a}\right) + \sqrt{\pm 2a(y+d) - (y+d)^2},$$

and differentiating three times, eliminating c, a and d. The actual tautochrones of this system are the cycloids concave upwards corresponding to the upper signs in the above equation. The system of straight lines in the plane also belongs to the totality of tautochrones represented by equation (6). A straight line may be regarded as a tautochrone whose center is at infinity.

3. We shall now seek geometric properties of the system (5) and determine the necessary and sufficient conditions that a triply infinite system of curves shall be a system of tautochrones.

Geometric Properties of the System.

4. *The focal curve.* Consider first the system of tautochrones through a given point (x, y) of the plane in a given direction y'. This is a system of ∞^1 curves, each corresponding to a different value of the time constant. Suppose that at the given point (x, y) the osculating parabola of each of these curves is constructed; that is, the parabola having four-point contact with the curve. Then with each of the ∞^1 curves of the system may be associated a point, namely, the focus of its osculating parabola constructed at the given point. The locus of the foci of these osculating parabolas is called the *focal curve.** We now determine the focal curve of the above system of tautochrones.

The coordinates α, β of the focus of an osculating parabola, referred to the point (x, y) as origin, are given by the formulas

$$\alpha = \frac{-3y''}{2} \frac{y'''(y'^2 - 1) + 2y'(3y''^2 - y'y''')}{y'''^2 + (3y''^2 - y'y''')^2}, \tag{7}$$

$$\beta = \frac{-3y''}{2} \frac{(3y''^2 - y'y''')(y'^2 - 1) - 2y'y'''}{y'''^2 + (3y''^2 - y'y''')^2}. \tag{8}$$

We now eliminate y'' and y''' between these equations (7) and (8) and the equation of the tautochrones (5). From equations (7) and (8),

$$\frac{\alpha}{\beta} = \frac{y'''(y'^2 - 1) + 2y'(3y''^2 - y'y''')}{(3y''^2 - y'y''')(y'^2 - 1) - 2y'y'''}. \tag{9}$$

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* Kasner: "The Trajectories of Dynamics," *Transactions of the American Mathematical Society*, Vol. VII (1906), p. 405.

The elimination of y''' between equations (8) and (9) gives

$$y'' = -\frac{(\alpha y' - \beta)(1 + y'^2)}{2(\alpha^2 + \beta^2)}. \tag{10}$$

Substituting this value of y'' in (5), and the resulting value of y''' in (9), we obtain, after reduction,

$$\left.\begin{array}{l} 4[\phi_{xx} + (2\phi_{xy} + \psi_{xx})y' + (\phi_{yy} + 2\psi_{xy})y'^2 + \psi_{yy}y'^3](\alpha^2 + \beta^2)^2 \\ \quad - 2[\phi_y + 2\psi_x + 3(\psi_y - \phi_x)y' - (2\phi_y + \psi_x)y'^2](\alpha y' - \beta)(\alpha^2 + \beta^2) \\ \quad - (\alpha y' - \beta)[(\psi y'^2 - 2\phi y' + 3\psi)\alpha - (3\phi y'^2 - 2\psi y' + \phi)\beta] = 0. \end{array}\right\} \tag{11}$$

This is the general equation of the focal curve, α and β being the coördinates of a point of the curve referred to the given point (x, y) as origin. The coefficients in equation (11) are determined by the coordinates x, y of the given point and the given direction y', since, for a positional force, ϕ and ψ are functions only of x and y.

An inspection of equation (11) shows that it represents a bicircular quartic having a node at the origin. The equations of the two tangents at the origin, t_1, t_2, say, are

$$t_1: \quad \alpha y' - \beta = 0, \tag{12}$$

$$t_2: \quad (\psi y'^2 - 2\phi y' + 3\psi)\alpha - (3\phi y'^2 - 2\psi y' + \phi)\beta = 0. \tag{13}$$

The tangent t_1 has contact of the third order with the quartic, cutting it at the origin in four coincident points, three on one branch and one on the other; it is, then, a stationary tangent. Its slope $\beta/\alpha = y'$. The other tangent t_2 has contact of the second order with the quartic, cutting it at the origin in three coincident points, two on one branch and one on the other. The origin is, then, a flecnode of the quartic, since one of the tangents is stationary.

THEOREM I. *At any point (x, y) of the plane in a given direction y', due to a positional force, there is a singly infinite system of tautochrones, each of which corresponds to a different value of the time constant. The focal curve of this system is a bicircular quartic having a flecnode at the origin, the stationary tangent having the given direction y'.*

The slope of the tangent t_2, from equation (13), is

$$\frac{\beta}{\alpha} = \frac{\psi y'^2 - 2\phi y' + 3\psi}{3\phi y'^2 - 2\psi y' + \phi}.$$

Taking the x axis in the direction y' of the tangent t_1, we have

$$\text{Slope of } t_2 = \frac{3\psi}{\phi};$$

but ψ/ϕ is the tangent of the angle which the direction of the acting force makes with the tangent t_1. Calling this angle θ' and the angle between t_1 and t_2, θ, we have

$$\tan\theta = 3\tan\theta'. \tag{14}$$

THEOREM II. *The second tangent t_2 to the quartic at the origin is such that the tangent of the angle between t_1 and t_2 is three times the tangent of the angle between t_1 and the direction of the acting force.*

5. *The focal curve for special forces.* When the acting force is gravity, then by substituting $\phi = 0$, $\psi = g$, in equation (11), the focal curve reduces to

$$(\alpha y' - \beta)\left[(y'^2 + 3)\alpha + 2y'\beta\right] = 0.$$

This represents two straight lines through the origin, and if θ is the angle between them, we find $\tan\theta = -3/y'$. When $y' = 0$, $\theta = 90°$ and the lines are the x and y axes. For actual tautochrones the focal curve consists only of a part of this locus, the upper half of the y axis.

When the coefficient of the fourth degree terms in equation (11) vanishes, the focal curve degenerates into the line t_1 and a circle through the origin tangent to the line t_2. This happens only when the force is that defined by the set of partial differential equations:

$$\left.\begin{array}{ll} \phi_{xx} = 0, & 2\phi_{xy} + \psi_{xx} = 0, \\ \phi_{yy} + 2\psi_{xy} = 0, & \psi_{yy} = 0. \end{array}\right\} \tag{15}$$

This class of forces is discussed by Kasner in the first paper cited; it is the class of forces for which every straight line in the plane is a tautochrone.

6. *The inverse curve.* We now take the inverse of the focal quartic with respect to the origin. Applying the formulas of inversion,

$$\alpha = \frac{\xi}{\xi^2+\eta^2}, \quad \beta = \frac{\eta}{\xi^2+\eta^2}, \quad \alpha^2 + \beta^2 = \frac{1}{\xi^2+\eta^2}, \tag{16}$$

to equation (11), we obtain

$$\left.\begin{array}{c} [(3\psi - 2\phi y' + \psi y'^2)y']\xi^2 - [(3\psi - 2\phi y' + \psi y'^2) + (\phi - 2\psi y' + 3\phi y'^2)y']\xi\eta \\ + [\phi - 2\psi y' + 3\phi y'^2]\eta^2 + 2[\phi_y + 2\psi_x + 3(\psi_y - \phi_x)y' - (2\phi_y + \psi_x)y'^2](\xi y' - \eta) \\ - 4[\phi_{xx} + (2\phi_{xy} + \psi_{xx})y' + (\phi_{yy} + 2\psi_{xy})y'^2 + \psi_{yy}y'^3) = 0, \end{array}\right\} \tag{17}$$

which is the equation of the inverse curve. Here, ξ, η are the coordinates of a point of the inverse curve referred to the given point (x, y) as origin, and the coefficients are determined when x, y and y' are given. The curve is a conic and its discriminant

$$[(3\psi - 2\phi y' + \psi y'^2) - (\phi - 2\psi y' + 3\phi y'^2) y']^2 > 0;$$

hence the conic is a hyperbola.

We find the equations of the asymptotes of the hyperbola to be

$$\eta = y' \xi, \tag{18}$$

$$\eta = \frac{3\psi - 2\phi y' + \psi y'^2}{\phi - 2\psi y' + 3\phi y'^2} \xi + \frac{2[\phi_y + 2\psi_x + 3(\psi_y - \phi_x) y' - (2\phi_y + \psi_x) y'^2)}{\phi - 2\psi y' + 3\phi y'^2}. \tag{19}$$

Hence one asymptote passes through the origin in the given direction y'; it is the tangent t_1 to the quartic [equation (12)]. The other asymptote has the same slope as the tangent t_2 to the quartic [equation (13)].

The inverse of the focal quartic is a hyperbola. One of its asymptotes is the tangent t_1 to the quartic at the origin; the other asymptote is parallel to the other tangent t_2 to the quartic at the origin.

7. *The locus of the center of the inverse curve.* So far we have considered only the system of ∞^1 tautochrones that pass through a given point in a given direction. For such a system we have found that Theorems I and II hold. Now let the tangent t_1, in the given direction y', rotate about the given point (x, y). For each position of t_1 we have a system of ∞^1 tautochrones, such as that of Article 4, passing through the given point and tangent to t_1. We thus obtain a system of ∞^2 curves—all the tautochrones in the plane passing through a given point. For each position of t_1 there is a focal quartic of the system of tautochrones tangent to t_1 at the given point. The curve inverse to this quartic is a hyperbola, the coordinates of whose center, referred to the point (x, y) as origin, we will denote by X and Y. Hence, with each direction y' of t_1 may be associated a point (X, Y) of the plane. We now determine the locus of the point (X, Y) as y' varies.

Since the center of the hyperbola is on the line t_1 whose slope is y', we have

$$Y = y' X. \tag{20}$$

Applying the formula,

$$X = \frac{bg - hf}{h^2 - ab},$$

for the abscissa of the center of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

to the conic (17), we get, after reduction,

$$X = -\frac{2\left[\phi_y + 2\psi_x + 3(\psi_y - \phi_x)y' - (2\phi_y + \psi_x)y'^2\right]y'}{(3\psi - 2\phi y' + \psi y'^2) - (\phi - 2\psi y' + 3\phi y'^2)y'}. \tag{21}$$

Eliminating y' between equations (20) and (21), we obtain

$$3(\psi X - \phi Y)(X^2 + Y^2) + 2\left[(\phi_y + 2\psi_x)X^2 + 3(\psi_y - \phi_x)XY - (2\phi_y + \psi_x)Y^2\right] = 0. \tag{22}$$

This represents the locus of the centers of the hyperbolas inverse to the focal curves, X, Y being the coordinates of a point on the locus referred to the given point (x, y) as origin.

An inspection of equation (22) shows that it represents a circular cubic having a double point at the origin. Working out the equation of the real asymptote of the cubic, we obtain

$$Y = \frac{\psi}{\phi}X + \frac{2\left[(\phi_y + 2\psi_x)\phi^2 + 3(\psi_y - \phi_x)\phi\psi - (2\phi_y + \psi_x)\psi^2\right]}{3\phi(\phi^2 + \psi^2)}. \tag{23}$$

The slope of this line is the same as the slope of the line of force at the given point. The asymptote is therefore parallel to the direction in which the force is acting at the given point.

THEOREM III. *Through a given point* (x, y) *of the plane, in a given direction* y', *passes a singly infinite system of tautochrones, the inverse of the focal curve of the system, with respect to the given point* (x, y), *being a hyperbola. The locus of the center of this hyperbola as* y' *varies is a circular cubic having the given point as double point. The real asymptote of the cubic is parallel to the direction of the force acting at the given point.*

8. *The tangents to the cubic at the origin.* From equation (22), the tangents to the cubic at the origin are given by

$$Y = \frac{3(\psi_y - \phi_x) \pm \sqrt{9(\psi_y - \phi_x)^2 + 4(2\phi_y + \psi_x)(\phi_y + 2\psi_x)}}{2(2\phi_y + \psi_x)}X. \tag{24}$$

These tangents will be real and distinct, coincident, or imaginary; that is, the origin will be a node, cusp, or conjugate point of the cubic according as

$$D = 9(\psi_y - \phi_x)^2 + 4(2\phi_y + \psi_x)(\phi_y + 2\psi_x) >, =, < 0. \tag{25}$$

If γ represents the angle between these tangents, we have

$$\tan\gamma = \frac{\sqrt{9(\psi_y - \phi_x)^2 + 4(2\phi_y + \psi_x)(\phi_y + 2\psi_x)}}{\phi_y - \psi_x}. \tag{26}$$

Special cases. For a conservative force, $\phi_y = \psi_x$. In this case (25) gives

$$D = 9(\psi_y - \phi_x)^2 + 36\phi_y^2 > 0.$$

The origin is a node, and from (26), $\gamma = 90°$. Conversely, when $\gamma = 90°$, $\phi_y = \psi_x$. This property is therefore characteristic of conservative forces.

A system of tautochrones will be a system due to a conservative force when and only when the tangents at the origin to the cubic of Theorem III are perpendicular to each other.

For a so-called analytic force,* $\phi_x = \psi_y$, $\phi_y = -\psi_x$. In this case (25) gives $D = -4\psi_x^2 < 0$. The origin is a conjugate point, and from (26), $\tan\gamma = i$. Conversely, if $\tan\gamma = i$, $\phi_x = \psi_y$ and $\phi_y = -\psi_x$; hence the property is characteristic of analytic forces.

9. *Hyperosculating circles.* Suppose that the circle of curvature of each tautochrone of the doubly infinite system through the point (x, y) is constructed at that point. Out of the ∞^2 circles thus constructed we now determine which ones hyperosculate their respective tautochrones. For hyperosculation the contact must be of the third order; hence the value of y''' in the equation of the circle must be equal to the value of y''' in the equation of the tautochrone.

The differential equation of all circles is

$$y''' = \frac{3y'y''^2}{1 + y'^2}. \tag{27}$$

Equating this value of y''' to the value of y''' in the equation of the tautochrones (5), and reducing, we find

$$\left.\begin{aligned} &(\phi + \psi y')y''^2 - [\phi_y + 2\psi_x + 3(\psi_y - \phi_x)y' - (2\phi_y + \psi_x)y'^2](1 + y'^2)y'' \\ &\quad - [\phi_{xx} + (2\phi_{xy} + \psi_{xx})y' + (\phi_{yy} + 2\psi_{xy})y'^2 + \psi_{yy}y'^3](1 + y'^2)^2 = 0, \end{aligned}\right\} \tag{28}$$

which is the condition for hyperosculating circles.

This is a quadratic in y'', regarding the point (x, y) and the direction y' as given. Hence there are two values of y'' for each lineal element in the plane, corresponding to which two tautochrones are hyperosculated by their circles of curvature. Thus, with each lineal element through a given point we may associate two points, namely the centers of the two hyperosculating circles determined by this element. We now inquire what will be the locus of these centers as the element rotates about the given point; that is, as y' varies in equation (28).

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* That is, a force whose rectangular components ϕ, ψ are conjugate harmonic functions, so named by Lecornu, "Sur les forces analytiques," *Journal de l'École Polytechnique*, Vol. LV (1885).

If we represent by $\bar{X}$ and $\bar{Y}$ the coordinates of the center of curvature referred to the given point (x, y) as origin, we have the formulas:

$$\bar{X} = -\frac{y'(1+y'^2)}{y''}, \quad \bar{Y} = \frac{1+y'^2}{y''}. \tag{29}$$

Substituting in equation (28) the values of y' and y'' found from these equations, we obtain

$$\left.\begin{aligned} &\phi_{xx}\,\bar{Y}^3 - (2\,\phi_{xy} + \psi_{xx})\,\bar{X}\,\bar{Y}^2 + (\phi_{yy} + 2\,\psi_{xy})\,\bar{X}^2\,\bar{Y} - \psi_{yy}\,\bar{X}^3 \\ &+ (\phi_y + 2\,\psi_x)\,\bar{Y}^2 - 3\,(\psi_y - \phi_x)\,\bar{X}\,\bar{Y} - (2\,\phi_y + \psi_x)\,\bar{X}^2 - \phi\,\bar{Y} + \psi\,\bar{X} = 0. \end{aligned}\right\} \tag{30}$$

This represents the locus of the centers of the hyperosculating circles, $\bar{X}$, $\bar{Y}$ being the coordinates of a point on the locus referred to the given point (x, y) as origin. The coefficients are determined by the coordinates of the given point. The curve is a cubic through the origin. The tangent at the origin is

$$\bar{Y} = \frac{\psi}{\phi}\,\bar{X}; \tag{31}$$

hence it is the line along which the force acts at that point.

THEOREM IV. *Corresponding to each lineal element through a given point there are two tautochrones tangent to the element at that point which are hyperosculated by their circles of curvature. The locus of the centers of these hyperosculating circles as the slope of the element changes is a cubic through the given point. The tangent to the cubic at the given point is the line along which the force acts at that point.*

10. We have found that the triply infinite system of tautochrones represented by equation (5) is such that the singly infinite system of them, passing through any point in the plane in a given direction, possesses the properties stated in Theorems I and II, and the doubly infinite system of them, passing through any point in the plane, possesses the properties stated in Theorems III and IV. We shall express this by saying that the triply infinite system possesses properties I, II, III and IV.

Converse Questions.

11. We have shown that the triply infinite system of tautochrones in the plane possesses properties I, II, III and IV. The question now arises, does any triply infinite system of curves possessing properties I–IV represent a family of tautochrones? We now proceed to convert these four properties in succession.

12. *Conversion of property I.* The focal curve of any system of curves having property I will be of the form

$$(\alpha^2+\beta^2)^2+A_1(\alpha y'-\beta)(\alpha^2+\beta^2)+(A_2\alpha+A_3\beta)(\alpha y'-\beta)=0. \qquad (32)$$

This is the general equation of a bicircular quartic having a flecnode at the origin, the stationary tangent at the origin having the direction y', where A_1, A_2, A_3 are arbitrary functions of x, y and y'.

We now eliminate α and β between equations (8) and (32) by means of relation (9). This gives

$$y'''=B_1+B_2y''+B_3y''^2, \qquad (33)$$

where B_1, B_2, B_3 involve A_1, A_2, A_3 and are functions of x, y and y'. Thus all systems possessing property I are of the form (33).

13. *Conversion of property II.* In order to convert Theorem II, we state it in a form not assuming a field of force, as follows: At any point (x, y) of the plane there exists a certain direction, whose slope is $\omega(x, y)$, such that the tangent of the angle between the two tangents t_1 and t_2 to the focal quartic at the given point is three times the tangent of the angle between t_1 and this direction. We now seek the equation of the system possessing property II in addition to property I.

The focal curve of the general system (33) possessing property I is

$$\left.\begin{aligned} 4B_1(\alpha^2+\beta^2)^2-2B_2(\alpha y'-\beta)(\alpha^2+\beta^2)\\ +(\alpha y'-\beta)\{[B_3y'(y'^2+1)-3(y'^2-1)]\alpha-[B_3(y'^2+1)-6y']\beta\}=0.\end{aligned}\right\} \qquad (34)$$

The slopes of the tangents t_1 and t_2 at the origin are respectively

$$\frac{\beta}{\alpha}=y', \quad \frac{B_3y'(y'^2+1)-3(y'^2-1)}{B_3(y'^2+1)-6y'}. \qquad (35)$$

If the system possesses property II, we then have

$$\frac{\dfrac{B_3 y'(y'^2+1)-3(y'^2-1)}{B_3(y'^2+1)-6y'}-y'}{1+\dfrac{B_3 y'(y'^2+1)-3(y'^2-1)}{B_3(y'^2+1)-6y'}y'}=\frac{3(\omega-y')}{1+\omega y'}. \tag{36}$$

Solving for B_3, we find

$$B_3=\frac{1+4\omega y'-3y'^2}{(\omega-y')(y'^2+1)}. \tag{37}$$

Hence all systems possessing both properties I and II have the form

$$y'''=B_1+B_2 y''+\frac{1+4\omega y'-3y'^2}{(\omega-y')(1+y'^2)}y''^2. \tag{38}$$

14. *Conversion of property III.* We may state property III in a form not assuming a field of force, as follows: Corresponding to each lineal element through a given point there is a focal quartic whose inverse with respect to the given point is a hyperbola. The locus of the center of this hyperbola as y' varies is a circular cubic having the given point as double point; the real asymptote of the cubic is parallel to the direction ω introduced in property II. We now determine the equation of the system having property III in addition to properties I and II.

The focal curve of the system (38) may be obtained from (34) by substituting in it the value of B_3 given in (37). Then inverting this focal curve by means of formulas (16), we obtain the hyperbola

$$\left.\begin{aligned}&[(3\omega-2y'+\omega y'^2)y']\xi^2-[(3\omega-2y'+\omega y'^2)+(1-2\omega y'+3y'^2)y']\xi\eta\\&\quad+[1-2\omega y'+3y'^2]\eta^2-2B_2(\xi y'-\eta)(\omega-y')+4B_1(\omega-y')=0.\end{aligned}\right\} \tag{39}$$

If property III holds for this system, the center of the hyperbola (39) must lie on the cubic

$$(\omega X-Y)(X^2+Y^2)+H_1X^2+H_2XY+H_3Y^2=0, \tag{40}$$

where H_1, H_2 and H_3 are arbitrary functions of x and y. Computing the coordinates of the center of the hyperbola, making use of relation (20), we find

$$X=\frac{2B_2}{3(1+y'^2)},\quad Y=\frac{2B_2 y'}{3(1+y'^2)}. \tag{41}$$

Substituting in (40), we get

$$B_2 = -\frac{3(H_1 + H_2 y' + H_3 y'^2)}{2(\omega - y')} = \frac{L_1 + L_2 y' + L_3 y'^2}{\omega - y'}, \tag{42}$$

where L_1, L_2 and L_3 are functions of x and y. Hence all systems having properties I, II and III are of the form

$$y''' = B_1 + \frac{L_1 + L_2 y' + L_3 y'^2}{\omega - y'} y'' + \frac{1 + 4\omega y' - 3y'^2}{(\omega - y')(1 + y'^2)} y''^2. \tag{43}$$

15. *Conversion of property IV.* Finally, we inquire what systems possess property IV in addition to properties I, II and III. Property IV may be stated without reference to force, as follows: For a system of curves through a given point in the plane, the locus of the centers of the hyperosculating circles constructed at the given point is a cubic through that point, whose tangent at the given point has the slope ω introduced in property II.

If the curves of the system (43) are to possess property IV, the centers of their hyperosculating circles must lie on the cubic,

$$M_1\bar{Y}^3 + M_2\bar{X}\bar{Y}^2 + M_3\bar{X}^2\bar{Y} + M_4\bar{X}^3 + N_1\bar{Y}^2 + N_2\bar{X}\bar{Y} + N_3\bar{X}^2 + \bar{Y} - \omega\bar{X} = 0, \tag{44}$$

where the M's and N's are functions of x and y. Equating the values of y''' in equations (43) and (27) gives the equation of condition for hyperosculation. If, in this equation, we substitute the values of y' and y'' found from (29), we obtain the locus of the centers of the hyperosculating circles for system (43), namely

$$\frac{B_1(\omega\bar{Y} + \bar{X})\bar{Y}^4}{\bar{X}^2 + \bar{Y}^2} + L_1\bar{Y}^2 - L_2\bar{X}\bar{Y} + L_3\bar{X}^2 + \bar{Y} - \omega\bar{X} = 0. \tag{45}$$

If this equation is to be of the form (44), we find

$$B_1 = \frac{(K_1 + K_2 y' + K_3 y'^2 + K_4 y'^3)(1 + y'^2)}{\omega - y'}, \tag{46}$$

where the K's are functions of x and y; so that we have finally this result:

The general equation of the system of curves having properties I, II, III and IV is

$$\left.\begin{aligned} y''' = {} & \frac{(K_1 + K_2 y' + K_3 y'^2 + K_4 y'^3)(1 + y'^2)}{\omega - y'} \\ & + \frac{L_1 + L_2 y' + L_3 y'^2}{\omega - y'} y'' + \frac{1 + 4\omega y' - 3y'^2}{(\omega - y')(1 + y'^2)} y''^2. \end{aligned}\right\} \tag{47}$$

This equation contains eight arbitrary functions of x and y, namely K_1, K_2, K_3, K_4, L_1, L_2, L_3 and ω. It is of the same form in y', y'' and y''' as equation (5), but more general with respect to the functions of x and y which enter into the coefficients. Properties I–IV are, then, not sufficient to characterize the total triply infinite system of tautochrones in the plane (5), since these properties belong to the more general system (47). In the next section we shall find the relations which must hold between the eight functions K_1, K_2, K_3, K_4, L_1, L_2, L_3 and ω for a complete characterization.

If, however, we consider only the doubly infinite system of curves passing through a point (x, y) and possessing properties I–IV, the curves of this system will have at that point the same values of y', y'' and y''' as the curves of the doubly infinite system of tautochrones passing through that point; that is, properties I–IV characterize the system of tautochrones through a given point up to differential elements of the third order. This can be shown in the following way:

For a given point, the eight arbitrary functions of x and y in equation (47) reduce to constants. If, now, these eight functions are chosen as arbitrary constants, it is required to find functions ϕ, ψ which for a given value of x and y make equation (5) equivalent to equation (47); that is, we must find functions ϕ, ψ which satisfy the following set of equations obtained by equating the coefficients in equations (5) and (47):

$$\left.\begin{aligned} K_1 &= -\frac{\phi_{xx}}{\phi}, & L_1 &= \frac{\phi_y + 2\psi_x}{\phi}, \\ K_2 &= -\frac{2\phi_{xy} + \psi_{xx}}{\phi}, & L_2 &= \frac{3(\phi_x - \psi_y)}{\phi}, \\ K_3 &= -\frac{\phi_{yy} + 2\psi_{xy}}{\phi}, & L_3 &= \frac{2\phi_y + \psi_x}{\phi}, \\ K_4 &= -\frac{\psi_{yy}}{\phi}, & \omega &= \frac{\psi}{\phi}, \end{aligned}\right\} \quad (48)$$

in which K_1, K_2, K_3, K_4, L_1, L_2, L_3 and ω are arbitrary constants. Let

$$\left.\begin{aligned} \phi &= a + bx + cy + dxy + ex^2 + fy^2, \\ \psi &= a' + b'x + c'y + d'xy + e'x^2 + f'y^2. \end{aligned}\right\} \quad (49)$$

Then for a given point, say $x = y = 0$, we have

$$\left.\begin{array}{llll} \phi = a, & \psi = a', & \phi_x = b, & \phi_y = c, \\ \psi_x = b' & \psi_y = c', & \phi_{xx} = 2e, & \phi_{xy} = d, \\ \phi_{yy} = 2f, & \psi_{xx} = 2e' & \psi_{xy} = d', & \psi_{yy} = 2f'. \end{array}\right\} \quad (50)$$

Then equations (48) reduce to

$$\left.\begin{array}{ll} a K_1 = -2e, & a L_1 = -(c + 2b'), \\ a K_2 = -2(d + e'), & a L_2 = 3(b - c'), \\ a K_3 = -2(f + d'), & a L_3 = 2c + b', \\ a K = -2f', & a\omega = a'. \end{array}\right\} \quad (51)$$

Take $a = 1$, $c' = d' = e' = 0$. Then

$$\left.\begin{array}{llll} e = -K_1/2, & d = -K_2/2, & f = -K_3/2, & f' = -K_4/2, \\ c = (L_1 + 2L_3)/3, & b = L_2/3, & b' = -(2L_1 + L_3)/3, & a' = \omega. \end{array}\right\} \quad (52)$$

Hence, from (49) and (52),

$$\left.\begin{array}{l} \phi = 1 + \frac{L_3}{2}x + \frac{L_1 + 2L_3}{3}y - \frac{K_2}{2}xy - \frac{K_1}{2}x^2 - \frac{K_3}{2}y^2, \\ \psi = \omega - \frac{2L_1 + L_3}{3}x - \frac{K_4}{2}y^2. \end{array}\right\} \quad (53)$$

The system of curves, possessing properties I–IV and passing through the point (0, 0), coincides up to differential elements of the third order with the system of tautochrones due to a force whose components are given by (53). The particular pair of functions (53) is only one of many pairs that satisfy the requirement. Properties I–IV are therefore sufficient to characterize the doubly infinite system of tautochrones through a given point up to differential elements of the third order.

Complete Characterization.

16. If the triply infinite system of curves represented by equation (47) is to be a system of tautochrones, it must be identical with the system represented by equation (5). The set of relations (48) must therefore hold, where now K_1, K_2, K_3, K_4, L_1, L_2, L_3 and ω represent arbitrary functions of x and y.

If we introduce

$$h = \log \phi, \quad (54)$$

and substitute

$$\psi = \omega\phi \quad (55)$$

in the equations for L_1, L_2 and L_3 in (48), we find

$$L_1 = -h_y - 2\omega h_x - 2\omega_x, \tag{56}$$

$$L_2 = 3(h_x - \omega h_y - \omega_y), \tag{57}$$

$$L_3 = 2h_y + \omega h_x + \omega_x. \tag{58}$$

From (56) and (58),

$$h_x = -\frac{3\omega_x + 2L_1 + L_3}{3\omega}, \quad h_y = \frac{L_1 + 2L_3}{3}. \tag{59}$$

Substituting these values in (57), we get the relation

$$\omega^2(L_1 + 2L_3) + \omega L_2 + 3(\omega_x + \omega\omega_y) + 2L_1 + L_3 = 0. \tag{A}$$

Also, from (59),

$$(L_1 + 2L_3)_x + \left(\frac{3\omega_x + 2L_1 + L_3}{\omega}\right)_y = 0. \tag{B}$$

By virtue of these two relations (A) and (B), we can express L_1, L_2, L_3 and ω in terms of two arbitrary functions ϕ and ψ; for, from (B) we can find a function h satisfying equations (59), thus giving (56) and (58); then (57) follows from (A), and, making the substitutions (54) and (55), L_1, L_2, L_3 and ω are obtained in the form given in (48).

Again, from (48),

$$\phi K_2 = -(\phi L_3)_x, \quad \phi K_3 = (\phi L_1)_y, \tag{60}$$

from which we get the two relations giving K_2 and K_3:

$$K_2 = -(L_3)_x + \frac{L_3(3\omega_x + 2L_1 + L_3)}{3\omega}, \tag{C}$$

$$K_3 = (L_1)_y + \frac{L_1(L_1 + 2L_3)}{3}. \tag{D}$$

To obtain the equations determining K_1, we make use of h_{xx}, found from (55),

$$h_{xx} = \frac{\phi_{xx}}{\phi} - h_x^2. \tag{61}$$

Then, from (61) and (59), the equation for K_1 (48) gives

$$K_1 = -\left(\frac{3\omega_x + 2L_1 + L_3}{3\omega}\right)^2 + \left(\frac{3\omega_x + 2L_1 + L_3}{3\omega}\right)_x. \tag{E}$$

Finally, to obtain the equation for K_4, from (49) we find the relation

$$\frac{\psi_y}{\phi} = \omega h_y + \omega_y. \tag{62}$$

Differentiating both sides of this equation with respect to y, and substituting for ψ_{yy}/ϕ its value, $-K_4$, from (48), we get

$$K_4 = -\omega\left(\frac{L_1 + 2L_3}{3}\right)^2 - 2\omega_y\frac{L_1 + 2L_3}{3} - \frac{\omega\left[(L_1)_y + 2(L_3)_y\right]}{3} - \omega_{yy}. \tag{F}$$

If, then, the relations (C), (D), (E) and (F) hold in addition to (A) and (B), they are sufficient for determining, in terms of two arbitrary functions ϕ and ψ, the values of K_1, K_2, K_3, K_4, L_1, L_2, L_3 and ω given in equations (48). We then have the final result:

Any triply infinite system of curves in the plane represents a family of tautochrones when, and only when, it possesses the properties I, II, III and IV, and the functions K_1, K_2, K_3, K_4, L_1, L_2, L_3 and ω, which appear in the equation (47) *of the system, satisfy the relations (A), (B), (C), (D), (E) and (F).*

Point Transformations.

17. *The formulas of transformation.* We shall refer to systems of curves possessing properties I; I and II; I, II, and III; and I, II, III and IV as systems of type I, II, III and IV respectively. The equations of these systems are respectively (33), (38), (43) and (47). We now consider the effect of an arbitrary point transformation on such systems.

We write the transformation formulas as follows, using the notation of Kasner,*

$$X = \Phi(x, y), \qquad Y = \Psi(x, y), \tag{63}$$

$$(cy' + d)\,Y' = ay' + b, \tag{64}$$

where

$$a = \Psi_y, \quad b = \Psi_x, \quad c = \Phi_y, \quad d = \Phi_x.$$

$$(cy' + d)^3\,Y'' = \alpha + \beta y' + \gamma y'^2 + \delta y'^3 + j y'', \tag{65}$$

where

$$\begin{aligned}
j &= ad - bc,\\
\alpha &= \Phi_x\Psi_{xx} - \Psi_x\Phi_{xx},\\
\beta &= \Phi_y\Psi_{xx} - \Psi_y\Phi_{xx} + 2\Phi_x\Psi_{xy} - 2\Psi_x\Phi_{xy},\\
\gamma &= \Phi_x\Psi_{yy} - \Psi_x\Phi_{yy} + 2\Phi_y\Psi_{xy} - 2\Psi_y\Phi_{xy},\\
\delta &= \Phi_y\Psi_{yy} - \Psi_y\Phi_{yy}.
\end{aligned}$$

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* Kasner: "Trajectories," *loc. cit.*, p. 419.

We shall use (65) in the abbreviated form

$$Y'' = \varepsilon_0 + \varepsilon_1 y''. \tag{65'}$$

$$\left.\begin{aligned}(c y' + d)^5 Y''' = (c y' + d)(A_0 + A_1 y' + A_2 y'^2 + A_3 y'^3 + A_4 y'^4)\\ - 3(\alpha + \beta y' + \gamma y'^2 + \delta y'^3)(\alpha_2 + \beta_2 y' + \gamma_2 y'^2)\\ + (\alpha_1 + \beta_1 y' + \gamma_1 y'^2) y'' - 3 c j y''^2 + j(c y' + d) y''',\end{aligned}\right\} \tag{66}$$

where

$$\alpha_1 = d(\beta + j_x) - 3 c \alpha - 3 j d_x,$$
$$\beta_1 = c(\beta + j_x) + d(2\gamma + j_y) - 3 c \beta - 3 j (c_x + d_y),$$
$$\gamma_1 = c(2\gamma + j_y) + 3 d \delta - 3 c \gamma - 3 j c_y,$$
$$\alpha_2 = \Phi_{xx}, \quad \beta_2 = 2\Phi_{xy}, \quad \gamma_2 = \Phi_{yy},$$
$$A_0 = \alpha_x, \quad A_1 = \alpha_y + \beta_x, \quad A_2 = \beta_y + \gamma_x, \quad A_3 = \gamma_y + \delta_x, \quad A_4 = \delta_y.$$

We shall use (66) in the abbreviated form

$$Y''' = E_0 + E_1 y'' + E_2 y''^2 + E_3 y'''. \tag{66'}$$

18. *Transformation of systems of type I.* If we apply the above transformation to the equation of systems of type I,

$$y''' = B_1 + B_2 y'' + B_3 y''^2, \tag{33}$$

it becomes

$$y''' = f_1 + f_2 y'' + f_3 y''^2, \tag{67}$$

where

$$f_1 = \frac{\overline{B}_1 + \overline{B}_2 \varepsilon_0 + \overline{B}_3 \varepsilon_0^2 - E_0}{E_3}, \quad f_2 = \frac{\overline{B}_2 \varepsilon_1 + 2\overline{B}_3 \varepsilon_0 \varepsilon_1 - E_1}{E_3}, \quad f_3 = \frac{\overline{B}_3 \varepsilon_1^2 - E_2}{E_3},$$

the bars denoting the result of applying the transformation to the quantities under them. The f's, like the B's, are functions of x, y and y', so that (67) is of the same form as (33).

Systems of type I are converted into systems of the same type by the general point transformation.

19. *Transformation of systems of type II.* For a system of type II, equation (38), we have

$$B_3 = \frac{1 + 4\omega y' - 3 y'^2}{(\omega - y')(1 + y'^2)}.$$

If this system is to remain of the same type, we must have

$$f_3 = \frac{\overline{B}_3 \varepsilon_1^2 - E_2}{E_3} = \frac{1 + 4\Omega y' - 3y'^2}{(\Omega - y')(1 + y'^2)}, \tag{68}$$

where Ω is a function of x and y.

Applying the transformation of Article 16 to B_3, we find

$$\overline{B}_3 = \frac{(cy' + d)\,[(cy' + d)^2 + 4\bar{\omega}(ay' + b)(cy' + d) - (ay' + b)^2]}{[(cy' + d)\bar{\omega} - (ay' + b)]\,[(ay' + b)^2 + (cy' + d)^2]}. \tag{69}$$

Then the value of f_3 becomes

$$f_3 = \frac{N}{D}, \tag{70}$$

$$\begin{aligned} N = {} & (4abd - b^2c + 3cd^2)\bar{\omega} - (4bcd - ad^2 + 3ab^2) \\ & + 2[(2a^2d + abc + 3c^2d)\bar{\omega} - (2bc^2 + acd + 3a^2b)]\,y' + 3(c\bar{\omega} - a)(a^2 + c^2)\,y'^2, \\ D = {} & [d\bar{\omega} - b + (c\bar{\omega} - a)\,y']\,[b^2 + d^2 + 2(ab + cd)\,y' + (a^2 + c^2)\,y'^2]. \end{aligned}$$

If this is to be of the form (63), we must have

$$ab + cd = 0, \quad a^2 + c^2 = b^2 + d^2, \tag{71}$$

in which case (70) reduces to

$$\left.\begin{aligned} f_3 &= \frac{1 + 4\Omega y' - 3y'^2}{(\Omega - y')(1 + y'^2)}, \\ \Omega &= \frac{d\bar{\omega} - b}{a - c\bar{\omega}}. \end{aligned}\right\} \tag{72}$$

Equations (71), written in the Φ, Ψ notation, are

$$\left.\begin{aligned} \Phi_x \Phi_y + \Psi_x \Psi_y &= 0, \\ \Phi_x^2 + \Psi_x^2 &= \Phi_y^2 + \Psi_y^2. \end{aligned}\right\} \tag{71'}$$

Eliminating Ψ_y, we find

$$(\Phi_x^2 + \Psi_x^2)(\Phi_y^2 - \Psi_x^2) = 0.$$

Disregarding the factor $\Phi_x^2 + \Psi_x^2$, since it leads only to imaginary solutions or to the trivial solution, $\Phi_x = \Psi_x = \Phi_y = \Psi_y = 0$, which makes Φ and Ψ constants and throws the whole plane into a single point, we have for the solutions of equations (71'),

$$\left.\begin{aligned} \Phi_x &= \Psi_y, \\ \Phi_y &= -\Psi_x, \end{aligned} \quad \text{or} \quad \begin{aligned} \Phi_x &= -\Psi_y, \\ \Phi_y &= \Psi_x. \end{aligned}\right\} \tag{73}$$

These are respectively the direct and inverse conformal transformations.

The only point transformations under which systems of type II are converted into systems of the same type are the conformal transformations.

20. *Transformation of systems of types III and IV.* For a system of type III, equation (43), we have

$$B_2 = \frac{L_1 + L_2 y' + L_3 y'^2}{\omega - y'}.$$

If this system is to remain of the same type, we must have, in addition to (68),

$$f_2 = \frac{B_3 \varepsilon_1 + 2 B_3 \varepsilon_0 \varepsilon_1 - E_1}{E_3} = \frac{M_1 + M_2 y' + M_3 y'^2}{\Omega - y'}, \tag{74}$$

where M_1, M_2 and M_3 are functions of x and y.

Under the transformation of Article 17, B_2 becomes

$$\overline{B}_2 = \frac{\overline{L}_1 (c y' + d)^2 + \overline{L}_2 (a y' + b)(c y' + d) + \overline{L}_3 (a y' + b)^2}{(c y' + d)\left[(c y' + d)\overline{\omega} - (a y' + b)\right]}.$$

It is found, by forming the expression for f_2 and imposing on it the condition (74), that this leads to the necessary condition

$$\alpha = \beta = \gamma = \delta = 0. \tag{75}$$

Condition (75), expressed in the Φ, Ψ notation, is

$$\left.\begin{aligned} &\Phi_x \Psi_{xx} - \Psi_x \Phi_{xx} = 0, \\ &\Phi_y \Psi_{xx} - \Psi_y \Phi_{xx} + 2 \Phi_x \Psi_{xy} - 2 \Psi_x \Phi_{xy} = 0, \\ &\Phi_x \Psi_{yy} - \Psi_x \Phi_{yy} + 2 \Phi_y \Psi_{xy} - 2 \Psi_y \Phi_{xy} = 0, \\ &\Phi_y \Psi_{yy} - \Psi_y \Phi_{yy} = 0. \end{aligned}\right\} \tag{75'}$$

The solution of this system of equations is*

$$\Phi = \frac{a_1 x + b_1 y + c_1}{a_3 x + b_3 y + c_3}, \quad \Psi = \frac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3}. \tag{76}$$

This is the projective transformation. If, then, a system of type III is to be converted into a system of the same type, the transformation must be projective as well as conformal. The most general transformation of this kind is the similitude transformation,

$$\begin{aligned} \Phi &= a x + b y + c, \\ \Psi &= \mp b x \pm a y + d, \end{aligned} \tag{77}$$

where a, b, c and d are constants. Under this transformation the condition (74) is satisfied. It is also found, by computing $\overline{B}_1$ and f_1, that systems of type IV remain of the same type under the transformation (77), so that we have finally the result:

---

* Lie-Scheffers: "Vorlesungen über continuierliche Gruppen," p. 84.

The only point transformations which convert systems of type III and IV into systems of the same type are the four-parameter group of similitude transformations.

Systems of Tautochrones for a Given k.

21. *The differential equation of the system.* The total triply infinite family of tautochrones in the plane may be grouped into doubly infinite systems according to different values of the time constant k. From equation (1) we have for the equation of such a system

$$T_s = k. \tag{78}$$

For each value of k this equation represents a doubly infinite system of curves.

From equation (4) we find

$$T_s = \overline{T} + \frac{N}{r}, \tag{79}$$

where

$$\overline{T} = \frac{\phi_x + (\phi_y + \psi_x)\, y' + \psi_y y'^2}{1 + y'^2}.$$

Hence the differential equation of the system (73), expressed in Cartesian form, is

$$\frac{\phi_x + (\phi_y + \psi_x)\, y' + \psi_y y'^2}{1 + y'^2} + \frac{(\psi - y'\phi)y''}{(1 + y'^2)^2} = k. \tag{80}$$

22. *Geometric properties of the system.* At every point (x, y) of the plane in a given direction y' and corresponding to a given value of k there is one tautochrone. Suppose that at the given point (x, y) the osculating circle of this tautochrone is constructed. Let α, β denote the coordinates of the center of this osculating circle, referred to the point (x, y) as origin. We inquire what will be the locus of this point (α, β) as y' varies.

The formulas for the coordinates of the center of curvature are

$$\alpha = -\frac{y'(1 + y'^2)}{y''}, \quad \beta = \frac{1 + y'^2}{y''}. \tag{81}$$

Eliminating y' and y'' between equations (80) and (81), we obtain

$$\alpha^2 (k - \psi_y) + \beta^2 (k - \phi_x) + (\phi_y + \psi_x)\,\alpha\beta - \phi\alpha - \psi\beta = 0. \tag{82}$$

This equation represents the locus of the centers of the osculating circles drawn at the point (x, y) to the tautochrones which pass through this point and correspond to a given value of the time constant k, α and β being the coordinates of a point of the locus referred to the given point (x, y) as origin. The locus is a conic passing through the given point. The tangent to the conic at the given point is

$$\beta = -\frac{\phi}{\psi}\alpha; \tag{83}$$

hence its direction is perpendicular to the direction in which the force is acting at the point.

In the case of gravity, $\phi = 0$, $\psi = g$, and the conic reduces to

$$k(\alpha^2 + \beta^2) - \beta = 0, \tag{84}$$

a circle tangent to the α axis at the origin.

It is also found that as k varies, the locus of the center of the above conic is another conic through the given point (x, y). The line along which the force acts at the given point is tangent to the second conic.

Space Tautochrones.

23. *The differential equation of the system of tautochrones in three-dimensional space.* Suppose that a positional force $F(x, y, z)$ is acting at every point of three-dimensional space. We shall now consider the system of space tautochrones due to such a force. The physical property of the tautochrones, expressed in equation (1), yields as in the plane case,

$$T_{ss} = 0, \tag{85}$$

where T is the tangential component of the force F along the tautochrone. We wish to express this equation in Cartesian form.

Let N represent the normal component of the force F, and ϕ, ψ and χ its x, y and z components respectively. Denoting the direction cosines of the tangent and principal normal by α, β, γ and l, m, n respectively, we have the relations

$$\left.\begin{aligned} &\alpha, \beta, \gamma = dx, dy, dz/ds = 1, y', z'/\sqrt{1 + y'^2 + z'^2}, \\ &l, m, n = \frac{d^2x}{ds^2}, \frac{d^2y}{ds^2}, \frac{d^2z}{ds^2} \Big/ \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}. \end{aligned}\right\} \tag{86}$$

Then we find

$$T=\phi\alpha+\psi\beta+\chi\gamma=\frac{\phi+\psi y'+\chi z'}{\sqrt{1+y'^2+z'^2}},\tag{87}$$

$$N=\phi l+\psi m+\chi n=\left(\phi\frac{d^2x}{ds^2}+\psi\frac{d^2y}{ds^2}+\chi\frac{d^2z}{ds^2}\right)r,\tag{88}$$

where r, the radius of absolute curvature, is

$$r=1\Big/\sqrt{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2}.$$

From equation (88) we get by means of relations (86)

$$\frac{N}{r}=\frac{\psi y''+\chi z''}{1+y'^2+z'^2}-\frac{(\phi+\psi y'+\chi z')(y'y''+z'z'')}{(1+y'^2+z'^2)^2}.\tag{89}$$

Differentiating equation (87) with respect to s, we find, after reduction,

$$T_s=\bar{T}+\frac{N}{r},\tag{90}$$

where

$$\bar{T}=\frac{\phi_x+\phi_y y'+\phi_z z'+(\psi_x+\psi_y y'+\psi_z z')y'+(\chi_x+\chi_y y'+\chi_z z')z'}{1+y'^2+z'^2}.\tag{91}$$

Differentiating equation (90) with respect to s gives the intrinsic equation of the tautochrones

$$\bar{T}_s+\left(\frac{N}{r}\right)_s=0.\tag{92}$$

If the values of $\bar{T}_s$ and $\left(\frac{N}{r}\right)_s$ are found from equations (91) and (89) by differentiation and substituted in (92), we obtain finally

$$Hy'''+Kz'''=A+By''+Cz''+Dy''z''+Ey''^2+Fz''^2,\tag{93}$$

where

$$\begin{aligned}
H&=\phi y'-\psi(1+z'^2)+\chi y'z',\\
K&=\phi z'-\chi(1+y'^2)+\psi y'z',\\
A&=[\phi_{xx}+(2\phi_{xy}+\psi_{xx})y'+(2\phi_{xz}+\chi_{xx})z'+2(\phi_{yz}+\psi_{xz}+\chi_{xy})y'z'\\
&\quad+(\phi_{yy}+2\psi_{xy})y'^2+(\phi_{zz}+2\chi_{xz})z'^2+(\psi_{zz}+2\chi_{yz})y'z'^2\\
&\quad+(2\psi_{yz}+\chi_{yy})y'^2z'+\psi_{yy}y'^3+\chi_{zz}z'^3](1+y'^2+z'^2),\\
B&=\phi_y+2\psi_x+3(\psi_y-\phi_x)y'+(\chi_y+2\psi_z)z'-3(\phi_z+\chi_x)y'z'\\
&\quad-(2\phi_y+\psi_x)y'^2+(\phi_y+2\psi_x)z'^2+3(\psi_y-\chi_z)y'z'^2\\
&\quad-(2\chi_y+\psi_z)y'^2z'+(\chi_y+2\psi_z)z'^3,
\end{aligned}$$

$$
\begin{aligned}
C = {} & \phi_z + 2\chi_x + (\psi_z + 2\chi_y)\,y' + 3\,(\chi_z - \phi_x)\,z' - 3\,(\phi_y + \psi_x)\,y'z' \\
& + (\phi_z + 2\chi_x)\,y'^2 - (2\phi_z + \chi_x)\,z'^2 - (2\psi_z + \chi_y)\,y'z'^2 \\
& + 3\,(\chi_z - \psi_y)\,y'^2 z' + (\psi_z + 2\chi_y)\,y'^3,
\end{aligned}
$$

$$
D = -\frac{3\,(\psi z' + \chi y') - [5\,(\psi y' + \chi z') + 8\,\phi]\,y'z' + 3\,(\psi z'^3 + \chi y'^3)}{1 + y'^2 + z'^2},
$$

$$
E = \frac{3\,\phi y'^2 - \phi - 4\,\psi y' + (3\,\chi y'^2 - \chi)\,z' - (\phi + 4\,\psi y')\,z'^2 - \chi z'^3}{1 + y'^2 + z'^2},
$$

$$
F = \frac{3\,\phi z'^2 - \phi - 4\,\chi z' + (3\,\psi z'^2 - \psi)\,y' - (\phi + 4\,\chi z')\,y'^2 - \psi y'^3}{1 + y'^2 + z'^2}.
$$

Equation (93) is the differential equation of the totality of tautochrones in three-dimensional space, due to a positional force whose rectangular components are ϕ, ψ and χ. It represents a system of ∞^{∞} curves. When χ and z are absent, the equation becomes that of the plane system, equation (5).

24. *Condition that every straight line in space shall be a tautochrone.* In order that every straight line in space shall be a tautochrone, it is necessary and sufficient that the force be such that equation (93) is satisfied when $y'' = z'' = 0$. Hence the coefficient A must vanish identically. This condition leads to the following system of equations:

$$
\left.
\begin{aligned}
\phi_{xx} &= 0, & 2\,\phi_{xz} + \chi_{xx} &= 0, \\
\psi_{yy} &= 0, & \phi_{zz} + 2\,\chi_{xz} &= 0, \\
\chi_{zz} &= 0, & 2\,\psi_{yz} + \chi_{yy} &= 0, \\
2\,\phi_{xy} + \psi_{xx} &= 0, & \psi_{zz} + 2\,\chi_{yz} &= 0, \\
\phi_{yy} + 2\,\psi_{xy} &= 0, & \phi_{yz} + \psi_{xz} + \chi_{xy} &= 0.
\end{aligned}
\right\} \quad (94)
$$

The solution of this set of equations may be obtained without difficulty. From the first three we have

$$
\left.
\begin{aligned}
\phi &= f(y, z) + x\,F(y, z), \\
\psi &= g\,(x, z) + y\,G\,(x, z), \\
\chi &= h\,(x, y) + z\,H(x, y),
\end{aligned}
\right\} \quad (95)
$$

where f, g, h, F, G, H are unknown functions. The remaining equations of (94) serve to determine these functions and we obtain for the solution of the system (94),

$$
\left.
\begin{aligned}
\phi &= a + b\,x + c\,y + d\,z + e\,x\,y + f\,x\,z + g\,y\,z + h\,y^2 + i\,z^2, \\
\psi &= a' + b'\,x + c'\,y + d'\,z - h\,x\,y + f'\,x\,z + g'\,y\,z - e\,x^2 + i'\,z^2, \\
\chi &= a'' + b''\,x + c''\,y + d''\,z - (f' + g)\,x\,y - i\,x\,z - i'\,y\,z - f\,x^2 - g'\,y^2,
\end{aligned}
\right\} \quad (96)
$$

where the coefficients are constants.

The class of forces defined by equations (96) is the class for which every straight line in space is a tautochrone. Equations (96) involve twenty parameters. The class of collineations involves fifteen parameters. Hence the relation, pointed out by Kasner,* which exists between these two classes in the plane, does not hold in space of three dimensions.

If the class of forces defined by equations (96) is required to be conservative, then

$$\phi_y = \psi_x, \quad \phi_z = \chi_x, \quad \psi_z = \chi_y, \tag{97}$$

in which case all the second partial derivatives of ϕ, ψ and χ vanish; the force components have the linear form,

$$\left.\begin{aligned} \phi &= a + bx + cy + dz, \\ \psi &= a' + cx + c'y + d'z, \\ \chi &= a'' + dx + d'y + d''z, \end{aligned}\right\} \tag{98}$$

and the work function is

$$W = ax + a'y + a''z + cxy + dxz + d'yz + \frac{bx^2}{2} + \frac{c'y^2}{2} + \frac{d''z^2}{2}. \tag{99}$$

Hence the result stated in the paper cited above for plane fields of force holds in space of three dimensions: *The only conservative forces for which every straight line is a tautochrone are those in which the potential is a quadratic function of x, y and z.*

COLUMBIA UNIVERSITY, *January*, 1910.

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* Kasner: "Tautochrones," *loc. cit.*, p. 480.

The General Transformation Theory of Differential Elements.[*]

By Edward Kasner.

The simplest type of differential element is the lineal element of the plane, defined by (x, y, y') or (x, y, p). The general element transformation is obtained by setting x_1, y_1, p_1 equal respectively to three arbitrary functions of x, y, p. Contact transformations are those special element transformations which convert every union into a union, a union (*Verein*) consisting of the ∞^1 elements of a curve or point. The general transformation will usually disrupt a union; that is, the new ∞^1 elements will not be united. Our first result is that for an arbitrary transformation (not of the contact class) there exist precisely ∞^2 unions which are preserved, that is, converted into unions. It follows that if a transformation preserves more than a doubly infinite number of unions, it must be a contact transformation.

We consider next transformations of the direction elements of space (x, y, z, y', z'). Here, intermediate between the general case preserving ∞^4 unions, and the contact case preserving all unions, we find a type preserving ∞^∞ unions. These are solutions of a certain form of Monge equation of the second order which is characterized geometrically by a property equivalent to that stated in Meusnier's theorem on the curves of a surface. The intermediate type may be obtained in explicit form and involves three arbitrary functions.

The discussion of transformations of surface elements (x, y, z, p, q), taken up in Section III, is more complicated. In the most general case no two-dimensional unions (surfaces, curves or points) are preserved. In the contact type, of course, all unions are preserved. In between we find types preserving a single, double or triple infinity, and also a type where the system of unions involves an arbitrary function, being defined by a pair of equations in involution. The last type, which is the most interesting as being the nearest to the contact type, depends upon three arbitrary functions of five arguments. Among the special results, it is shown that if every point and every plane is converted into a union, the transformation is necessarily one of contact. It follows, for example, that no element transformations beyond those contained in Lie's fifteen-parameter group can convert every sphere into a sphere.

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[*] The main results of this paper were presented before the American Mathematical Society, April 30, 1904.

With respect to one-dimensional unions (strips), all transformations not of the contact type preserve a family involving two arbitrary functions.

In Section IV we consider briefly the differential elements of n-th order in the plane. Usually, no unions are preserved, but the number may vary in special cases from ∞^1 to ∞^{n+1}. If more than ∞^{n+1}, then all are preserved.

I. *Direction Elements in the Plane.*

The general transformation is of the form

$$x_1 = X(x, y, p), \quad y_1 = Y(x, y, p), \quad p_1 = P(x, y, p), \tag{1}$$

where the three functions are assumed to be uniform, to possess continuous partial derivatives, and to have a non-vanishing Jacobian.

Consider now two united elements (x, y, p) and $(x + dx,\ y + dy,\ p + dp)$, so that $dy - p\,dx = 0$. The condition that the transformed elements (x_1, y_1, p_1) and $(x_1 + dx_1,\ y_1 + dy_1,\ p_1 + dp_1)$, where

$$dx_1 = X_x\,dx + X_y\,dy + X_p\,dp, \text{ etc.,}$$

shall also be united is $dy_1 - p_1\,dx_1 = 0$, or

$$(Y_x - PX_x)\,dx + (Y_y - PX_y)\,dy + (Y_p - PX_p)\,dp = 0,$$

which becomes, on replacing dy by $p\,dx$,

$$\{Y_x + pY_y - P(X_x + pX_y)\}\,dx + \{Y_p - PX_p\}\,dp = 0. \tag{2}$$

If (1) is a contact transformation, all united elements remain united, so that (2) is satisfied identically; we then have the familiar necessary and sufficient conditions

$$Y_x + pY_y - P(X_x + pX_y) = 0, \quad Y_p - PX_p = 0. \tag{3}$$

Omitting this case, we may solve (2) for dp/dx, that is d^2y/dx^2 or y'', and write

$$y'' = \frac{P(X_x + pX_y) - (Y_x + pY_y)}{Y_p - PX_p}. \tag{4}$$

Hence if (1) *is not a contact transformation, there is for each element a unique united element such that the two transformed elements are also united.*

Our condition (4) is a differential equation of the second order

$$y'' = \omega(x, y, y'), \tag{4'}$$

and therefore defines a family of ∞^2 curves. In the particular case where $Y_p - PX_p$ vanishes, (4′) takes the form $y'' = \infty$ and defines the ∞^2 points of the plane.

For a general element transformation (1) (*not a contact transformation*), *there exists always a doubly infinite system of unions which are alone converted into unions. The differential equation of the system is* (4).

As an important corollary, we have:

If a transformation converts more than ∞^2 *unions into unions, the same must be true for all unions, and hence it is a contact transformation.*

Consider, as an application, the determination of the group of element transformations which convert every circle into a circle. Since the number of circles is ∞^3, it follows that the result is merely Lie's ten-parameter group of contact transformations.

On the other hand, the group of element transformations which convert straight lines into straight lines is larger than the corresponding group of contact transformations. The latter involves two arbitrary functions of two arguments, while the former involves an additional arbitrary function of three arguments. Since all differential equations of the second order are equivalent under contact transformation, a similar remark applies to all doubly infinite systems of curves. In case of the ∞^2 point unions, the element transformations are obviously

$$x_1 = \phi(x, y), \quad y_1 = \psi(x, y), \quad p_1 = P(x, y, p);$$

while the contact transformations are

$$x_1 = \phi(x, y), \quad y_1 = \psi(x, y), \quad p_1 = \frac{\psi_x + p\psi_y}{\phi_x + p\phi_y}.$$

We note finally that if a transformation converts every point and every straight line into a union, it must be of the contact category.

II. *Direction Elements in Space.*

A direction element in space is defined by five coordinates (x, y, z, y', z'); we employ also the notation (x, y, z, p, q) when more convenient.* The general transformation is

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* In the next section p, q will be used in the customary manner to denote partial derivatives of z with respect to x and y.

$$x_1 = X(x, y, z, p, q), \quad \ldots, \quad q_1 = Q(x, y, z, p, q). \tag{5}$$

Lie showed that the only transformations of this sort which convert *all* curves (or unions) into curves (or unions) are the extended point transformations. We inquire whether the general transformation (5) converts *some* unions into unions.

Using the two conditions for a union,

$$dy - p\,dx = 0, \quad dz - q\,dx = 0,$$

and proceeding as in I, we find

$$\left.\begin{aligned}(PX_p - Y_p)dp + (PX_q - Y_q)dq &= \{Y_x + pY_y + qY_z - P(X_x + pX_y + qX_z)\}\,dx,\\ (QX_p - Y_p)dp + (QX_q - Y_q)dq &= \{Z_x + pZ_y + qZ_z - Q(X_x + pX_y + qX_z)\}\,dx.\end{aligned}\right\} \tag{6}$$

Omitting the case of extended point transformations, we see that these will not both hold identically. Assuming first that the determinant

$$\begin{vmatrix} PX_p - Y_p & PX_q - Y_q \\ QX_p - Y_p & QX_q - Y_q \end{vmatrix} \tag{7}$$

does not vanish, we may solve (6) in the form

$$y'' = g(x, y, z, y', z'), \quad z'' = h(x, y, z, y', z'), \tag{6'}$$

and have then a pair of equations defining ∞^4 curves.

On the other hand, if the above determinant vanishes identically, but the equations (6) are not equivalent, it follows that dx must vanish and dp and dq are connected by a single relation

$$(PX_p - Y_p)\,dp + (PX_q - Y_q)\,dq = 0. \tag{8}$$

Thus, for any fixed values of x, y, z, we find q as a function of p and a constant of integration. Our unions are then of special type, consisting of elements with a common point forming a cone. The total number of these is as before ∞^4.

We must consider finally the case where the equations (6) are equivalent to each other; that is, where

$$\left\| \begin{matrix} PX_p - Y_p & PX_q - Y_q & Y_x + pY_y + qY_z - P(X_x + pX_y + qX_z) \\ QX_p - Y_p & QX_q - Y_q & Z_x + pZ_y + qZ_z - Q(X_x + pX_y + qX_z) \end{matrix} \right\| = 0. \tag{9}$$

The single equation to be satisfied is then of the form

$$Ey'' + Fz'' + G = 0, \tag{10}$$

where E, F, G involve only x, y, z, y', z'. Such a Monge equation of the second order is satisfied by a family of curves involving an arbitrary function; in fact, upon any surface there will be ∞^2 integral curves. It is easily shown from the linear form of (10) that the integral curves have the *Meusnier property*; namely, the osculating circles of those curves of the family which pass through a given point in a given direction generate a sphere. Conversely, (10) represents the most general family for which this is true.*

The case where conditions (9) hold thus leads to a type of element transformation intermediate between the completely arbitrary type and the contact type. Our result may be stated as follows:

All transformations (5) *of the direction elements of space may be classified with respect to the number of unions preserved into these types:*

Type 1°. *Extended point transformations. All unions preserved.*

Type 2°. *Transformations fulfilling conditions* (9), *but not extended point transformations. The family of unions preserved involves an arbitrary function: It is defined by a Monge equation of the second order of the form* (10), *characterized by the possession of the Meusnier property.*

Type 3°. *Transformations not fulfilling conditions* (9). *The number of unions preserved is* ∞^4.

The last is, of course, the general case, involving five arbitrary functions of three arguments. Type 1° involves three arbitrary functions of three arguments. The intermediate type 2° involves three arbitrary functions of five arguments: we may in fact take the functions X, Y, Z at random and still satisfy conditions (9), the functions P and Q being then in general† uniquely determined.

III. *Surface Elements of Space.*

A surface element (x, y, z, p, q) consists of a point (x, y, z) and an incident plane, the direction cosines of the normal being proportional to $p:q:-1$. The condition for united elements is

$$dz = p\,dx + q\,dy.$$

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* Cf. the author's note "The Inverse of Meusnier's Theorem," *Bull. Amer. Math. Soc.*, Vol. XIV (1908), pp. 461–465.

† Unless certain determinants vanish.

In connection with a surface $z=f(x, y)$, we employ the usual notation $p=f_x$, $q=f_y$, $r=f_{xx}$, $s=f_{xy}$, $t=f_{yy}$, so that

$$dp = r\,dx + s\,dy, \quad dq = s\,dx + t\,dy.$$

Applying now the general element transformation

$$x_1 = X(x, y, z, p, q), \quad \ldots, \quad q_1 = Q(x, y, z, p, q), \tag{11}$$

we have, as the condition that the transformed elements form a union,

$$dz_1 = p_1\,dx_1 + q_1\,dy,$$

or

$$\alpha\,dx + \beta\,dy + \gamma\,dz + \delta\,dp + \varepsilon\,dq = 0, \tag{12}$$

where

$$\left.\begin{array}{c}\alpha = Z_x - PX_x - QY_x, \quad \beta = Z_y - PX_y - QY_y, \quad \gamma = Z_z - PX_z - QY_z, \\ \delta = Z_p - PX_p - QY_p, \quad \varepsilon = Z_q - PX_q - QY_q.\end{array}\right\} \tag{12'}$$

Substituting the values of dz, dp, and dq given above, and equating the coefficients of dx and dy in the result to zero, we find the fundamental pair of equations.

$$Ar + Bs + C = 0, \quad As + Bt + D = 0, \tag{13}$$

where the coefficients are

$$A = \delta, \quad B = \varepsilon, \quad C = \alpha + pr, \quad D = \beta + qr, \tag{13'}$$

and thus involve only x, y, z, p, q.

If our element transformation (11) *converts any surfaces into unions, such surfaces must satisfy the pair of equations* (13).

In the case of contact transformations the equations become identities: the familiar conditions for a contact transformation are in fact

$$A = B = C = D = 0. \tag{14}$$

We exclude this case from the remaining discussion.

A pair of partial differential equations of the second order involving one function of two independent variables will in general be inconsistent. In special cases there may be a finite number of solutions; a solution involving one, two, three or four arbitrary constants; or finally, when the equations are in involution, the solution may involve an infinity of constants, that is, an arbitrary function.

Applying the general theory to our pair (13), we differentiate each of the equations with respect to x and to y, and form with the coefficients of the four resulting equations, which are linear in the partial derivatives of third order, the following matrix:

$$\begin{vmatrix} A & B & 0 & 0 & r\dfrac{dA}{dx}+s\dfrac{dB}{dx}+\dfrac{dC}{dx} \\ 0 & A & B & 0 & r\dfrac{dA}{dy}+s\dfrac{dB}{dy}+\dfrac{dC}{dy} \\ 0 & A & B & 0 & s\dfrac{dA}{dx}+t\dfrac{dB}{dx}+\dfrac{dD}{dx} \\ 0 & 0 & A & B & s\dfrac{dA}{dy}+t\dfrac{dB}{dy}+\dfrac{dD}{dy} \end{vmatrix}, \tag{15}$$

where, for example,

$$\frac{dA}{dx} \equiv A_x + pA_z + rA_p + sA_q, \quad \frac{dA}{dy} \equiv A_y + qA_z + sA_p + tA_q. \tag{15'}$$

The condition for involution is that all the determinants of fourth order in the above matrix shall vanish in virtue of (13). Only two independent relations are found, namely

$$\left.\begin{aligned} &CD(A_q-B_p)-BC(A_y+qA_z-D_p)+AD(B_x+pB_z-C_q)+AB(C_y-D_x+qC_z-pD_z)=0, \\ &(AC+BD)(A_q-B_p)-B^2(A_y+qA_z-D_p)+A^2(B_x+pB_z-C_q) \\ &\qquad\qquad\qquad\qquad +AB(B_y-A_x+qB_z-pA_z+C_p-D_q)=0. \end{aligned}\right\} \tag{16}$$

Our result is then as follows:

If an element transformation (11), *without being of the contact type, converts an infinitude of surfaces involving an arbitrary function into unions, then the pair of equations* (13) *will be in involution and hence relations* (16) *will hold.*

The type characterized by relations (16) is thus the nearest to the contact type, in the sense that it preserves the maximum number of unions, without preserving all. The manifold of surfaces is such that ∞^1 pass through an arbitrarily selected curve.

If any pair of equations of the second order possess more than ∞^4 solutions, they will, according to standard theory, have ∞^∞ solutions, and hence be in involution. In the case of our equations (13) we obtain a stronger result. We notice that the determinant formed from the first four columns of the matrix (15) (that is, from the coefficients of the third derivatives in the set of four linear equations referred to above) vanishes identically.* The assumption of ∞^4 common solutions requires that the four linear equations shall be consistent. In the present case this is impossible unless *all* the determinants in (15) vanish; that is, the case of involution. Therefore:

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* This means that the equations (13) always have one set of characteristics in common.

If an element transformation converts ∞^4 *surfaces into unions, the equations* (13) *are necessarily in involution; that is,* ∞^∞ *surfaces are so converted. The transformation thus belongs to the type described in the preceding theorem* (*or else it is of the contact type*).

We next consider transformations preserving a triply infinite system of unions. This type, which is of course more general than the preceding type, can be determined explicitly from general principles. We observe first that any ∞^3 unions (surfaces, curves or points) may be converted into any other ∞^3 unions by a contact transformation. In particular, any ∞^3 unions may be converted into points. The most general element transformation which converts every point union into itself, is obviously of the form

$$x_1 = x, \quad y_1 = y, \quad z_1 = z, \quad p_1 = P(x, y, z, p, q), \quad q_1 = Q(x, y, z, p, q), \tag{17}$$

involving two arbitrary functions P and Q. Denote this infinite group by G, and let Γ_1, Γ_2 denote any two contact transformations. Then

$$\Gamma_1 G \Gamma_2, \tag{18}$$

converts the ∞^3 unions which become points under Γ_1 into the ∞^3 unions which are obtained by applying Γ_2 to points. All transformations of the required type are included under the form (18). Remembering that a general contact transformation involves an arbitrary function of five arguments, we state our result as follows:

All element transformations which preserve as many as ∞^3 *unions are included in the form* $\Gamma_1 T \Gamma_2$, *where* Γ_1, Γ_2 *are arbitrary contact transformations, and* T *is any member of the group* (17) *which converts every point union into itself. The result involves four arbitrary functions of five arguments.*

Consider in particular the case where the ∞^3 unions are the points or the planes of space. If every point is to be converted into a union, equation (12) must be satisfied on substituting $dx = dy = dz = 0$. It follows that δ and ϵ must vanish; that is, in the notation of (13′), $A = 0$, $D = 0$. On the other hand, if every plane is to become a union, our fundamental equation (13) must be satisfied when $r = s = t = 0$. The requisite conditions are $C = 0$, $D = 0$. Remembering that the vanishing of the quantities A, B, C, D is the criterion for a contact transformation, we derive this result:

If a transformation of the surface elements of space converts both points and planes into unions, it must be a contact transformation.

This conclusion does not hold for two sets of ∞^3 unions selected at random. For it may happen that the two sets are part of a system of ∞^∞ unions defined by a pair of equations (13) in involution, in which case there will exist transformations not of the contact type. Similar remarks apply, of course, to systems of unions involving four or more parameters.

For certain systems, however, only contact transformations exist. Consider the ∞^4 spheres. Their differential equations are

$$pqr - (1 + p^2)s = 0, \quad (1 + q^2)s - pqt = 0. \tag{19}$$

If these are equivalent to a set of type (13), we must have $A = B = C = D = 0$; that is, a contact transformation. This result also follows directly from that stated above, since spheres include points and planes as limiting cases. Recalling the equivalence of spheres and straight lines, we have:

No element transformations beyond the contact type can convert every sphere, or every straight line, into a union.

In particular, the known fifteen-parameter group of sphere transformations includes all element transformations which turn spheres into spheres.

We return to the consideration of equation (12), defining all unions which are preserved under (11). In the general case where the union (of ∞^2 elements) is a surface, we obtain the partial differential equations (13). The case of a curve regarded as a union of ∞^2 elements may be treated as a limiting case of a surface. The direct treatment however is as follows:

For the surface elements of a curve whose point equations are $y = y(x)$, $z = z(x)$, we have

$$p = z' - qy'. \tag{20}$$

Substituting in (12), we obtain an identity in dx, dq giving two conditions

$$\left.\begin{aligned} \alpha + \beta y' + \gamma z' + \delta(z'' - qy'') &= 0, \\ \varepsilon - y'\delta &= 0. \end{aligned}\right\} \tag{21}$$

These equations are, on eliminating p by means of (20), to become identities in q. Usually no solutions will exist; for certain element transformations there may be a triply infinite system of curves or even an infinitude involving an arbitrary function, in which case the family is defined by a Monge equation $f(x, y, z, y', z') = 0$. A detailed discussion is rendered unnecessary by observing that, to the curves turned into unions by any given transformation T correspond surfaces preserved by the related transformation RT, where R denotes polarity with respect to a quadric surface.

Summarizing our results, we have the following

THEOREM. *With respect to the number of unions (of ∞^3 elements) which are converted into unions, transformations of the surface elements of space may be classified as follows:*

Type 1°. *Contact transformations. All unions preserved.*

Type 2°. *Transformations, other than of type* 1°, *preserving ∞^∞ unions. The unions are then defined by an involution system.*

Type 3°. *Transformations, not of the previous types, preserving ∞^3 unions.*

Type 4°. *Transformations preserving fewer than ∞^3 unions. The number may be ∞^2, ∞^1, a discrete number, or none at all, the last being the usual case.*

Type 1° is characterized by four conditions $A = B = C = D = 0$, and thus involves one arbitrary function of five arguments. Type 2° is characterized by the two conditions (16), and depends on three arbitrary functions. Type 3°, represented explicitly by (18), involves four arbitrary functions. Type 4°, being the general case, involves five arbitrary functions.*

We consider finally the effect of any transformation on one-dimensional unions. The general case of such a union consists of the ∞^1 surface elements touching a given surface along a given curve, thus forming a strip. A special case arises when the ∞^1 elements have a common point, thus enveloping a cone, which may in particular shrink to a line. The strips preserved must satisfy the equations

$$\alpha + \beta y' + \gamma z' + \delta p' + \varepsilon q' = 0, \qquad p + q y' - z' = 0, \tag{22}$$

where y, z, p, q are considered as unknown functions of x. We easily derive the following results:

The only transformations preserving all one-dimensional unions are the contact transformations.

For an arbitrary element transformation, the number of strips preserved is ∞^∞, involving two arbitrary functions.

IV. *Elements of Higher Order.*

Confining our attention to the plane, we consider the general transformation of differential elements of the n-th order

$$\left.\begin{aligned} \bar{x} = X(x, y, y', \ldots, y^{(n)}), \quad \bar{y} = Y(x, y, y', \ldots, y^{(n)}), \\ \bar{y}' = Y_1(x, y, y', \ldots, y^{(n)}), \ \ldots, \ \bar{y}^{(n)} = Y_n(x, y, y', \ldots, y^{(n)}). \end{aligned}\right\} \tag{23}$$

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* This remains true if we take only those transformations which do not preserve any unions.

The conditions for a union are

$$dy = y'\,dx, \quad dy' = y''\,dx, \quad \ldots, \quad dy^{(n-1)} = y^{(n)}\,dx.$$

If a curve is to be converted into a union, we find that it must satisfy the following differential equations of the $(n+1)$-th order:

$$y^{(n+1)} = \omega_k(x, y, y', \ldots, y^{(n)}) \quad (k = 1, 2, \ldots, n), \tag{24}$$

where

$$\omega_k \equiv \frac{\dfrac{dY_{k-1}}{dx} - Y_k \dfrac{dX}{dx}}{\dfrac{\partial Y_{k-1}}{\partial y^{(n)}} - Y_k \dfrac{\partial X}{\partial y^{(n)}}}. \tag{24'}$$

The equations (24) become identities only in the case of extended contact transformations. *In general the equations will be inconsistent, so that no unions will be preserved. In intermediate types there may be* ∞^j *curves, where* j *varies from* 1 *to* $n+1$ *inclusive.*

If a transformation (23) preserves more than ∞^{n+1} unions, it must preserve all and hence be of the contact type. The type nearest to the contact type, preserving ∞^{n+1} curves, may easily be determined explicitly in terms of three arbitrary functions. We note that with respect to the group (23) all differential equations of the $(n+1)$-th order are equivalent.

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